

Tensor product of filtered A_∞ -algebras

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Abstract

We define the tensor product of filtered A_∞ -algebras, establish some of its properties and give a partial description of the space of bounding cochains in the tensor product. Furthermore we show that in the case of classical A_∞ -algebras our definition recovers the one given by Markl and Shnider. We also give a criterion that implies that a given A_∞ -algebra is quasi-isomorphic to the tensor product of two subalgebras. This will be used in a sequel to prove a Künneth Theorem for the Fukaya algebra of a product of Lagrangian submanifolds.

1 Introduction

In this paper we study the tensor product of A_∞ -algebras. The case of classical (or flat) A_∞ -algebras, that is algebras with $\mathfrak{m}_0 = 0$, was studied by several authors, see [3], [8], [10] and [12]. We will focus on a class of curved A_∞ -algebras over the Novikov ring Λ_0 , that has not been considered yet, namely filtered A_∞ -algebras. These were introduced by Fukaya, Oh, Ohta and Ono in [5], to study the obstruction to the existence of Lagrangian Floer cohomology.

For classical A_∞ -algebras, it follows from the general theory of minimal models of operads (see [11]) that the tensor product of A_∞ -algebras exists. However this does not provide an explicit model for the tensor product. The first explicit construction of the tensor product of classical A_∞ -algebras was given by Saneblidze and Umble in [12]. They do it by constructing an explicit diagonal for the operad governing A_∞ -algebras, the associahedra. Later Markl and Shnider [10] gave a somewhat more conceptual construction of a diagonal which they claim coincides with the construction of Saneblidze-Umble. Markl and Shnider use a cubical decomposition of the associahedra and then apply the usual Alexander-Whitney diagonal to this cubic complex. Loday [8] gave another construction of a diagonal, this time using a simplicial decomposition of the associahedra. All these constructions are quasi-isomorphic but in fact it seems they give the exact same \mathfrak{m}_k operations. This has not been checked for $k \geq 6$.

We will take a different approach, first used in [7], that has the advantage of not using operads. This makes it easier to generalize to the filtered case. First we reduce the problem to the case of filtered dg-algebras, that is filtered A_∞ -algebras with $\mathfrak{m}_k = 0$ for $k \geq 3$. Given a (unital) filtered A_∞ -algebra A we show there is a filtered dg-algebra End_A quasi-isomorphic to A . For classical algebras this follows from the Yoneda embedding (see [14]), but for filtered algebras it is new. For filtered dg-algebras there is an obvious definition of tensor product, so we take

$$A \otimes_\infty B := End_A \otimes_{dg} End_B.$$

This definition is very natural and satisfies (up to quasi-isomorphisms) the usual properties of the tensor product of associative algebras. It might however seem a bit unsatisfactory as it does not give an A_∞ -algebra structure on the vector space $A \otimes B$. We can remedy this situation using the homological perturbation lemma to transfer the A_∞ -algebra structure on $A \otimes_\infty B$ to the vector space $A \otimes B$. This way we obtain an

explicit A_∞ -algebra structure on $A \otimes B$, described in Theorem 4.1. We summarize the main properties of the tensor product \otimes_∞ in

Theorem 1.1. *Let A_i and B_i be filtered A_∞ -algebras. We have the following:*

1. *If $A_1 \simeq A_2$ and $B_1 \simeq B_2$ are quasi-isomorphic, then $A_1 \otimes_\infty B_1 \simeq A_2 \otimes_\infty B_2$;*
2. *$A \otimes_\infty \mathbb{K} \simeq A$, where \mathbb{K} is the ground field;*
3. *$A_1 \otimes_\infty A_2 \simeq A_2 \otimes_\infty A_1$;*
4. *$A_1 \otimes_\infty (A_2 \otimes_\infty A_3) \simeq (A_1 \otimes_\infty A_2) \otimes_\infty A_3$;*
5. *If A_i are flat A_∞ -algebras then $A_1 \otimes_\infty A_2$ is quasi-isomorphic to the tensor product $A_1 \otimes_{SU} A_2$ defined by Markl and Shnider in [10].*

Our second main theorem gives a set of conditions under which a filtered A_∞ -algebra C is quasi-isomorphic to the tensor product of two subalgebras A and B . For associative algebras the basic condition for this to happen is that elements of A and B should commute in C . Our definition of commuting subalgebras can be thought of as a generalization of this to the A_∞ world. Although completely algebraic, this definition and the construction used to prove the following theorem arose in symplectic geometry, in the author's Ph.D. thesis [1].

Theorem 1.2. *Let (A, \mathfrak{m}^A) and (B, \mathfrak{m}^B) be commuting subalgebras of (C, μ) in the sense of Definitions 5.1 and 5.2. If $K : A \otimes B \rightarrow C$ defined as $K(a \otimes b) = (-1)^{|a|} \mu_{2,0}(a, b)$ is an injective map which induces an isomorphism on $\mu_{1,0}$ -cohomology then there is a (strict) quasi-isomorphism*

$$A \otimes_\infty B \simeq C.$$

The main application of this theorem is the proof of a Künneth Theorem for the Fukaya algebra. Let L be a Lagrangian submanifold of a symplectic manifold (M, ω) , the Fukaya algebra $\mathcal{F}(L)$ is a filtered A_∞ -algebra structure on the singular chain (or de Rham) complex of L , constructed in [5]. Given Lagrangian submanifolds $L_i \subset (M_i, \omega_i)$, consider the product Lagrangian $L_1 \times L_2 \subset (M_1 \times M_2, \omega_1 \oplus \omega_2)$. In [2], it is proved that $\mathcal{F}(L_1)$ and $\mathcal{F}(L_2)$ are commuting subalgebras of $\mathcal{F}(L_1 \times L_2)$. The above theorem then implies that $\mathcal{F}(L_1 \times L_2)$ is quasi-isomorphic to $\mathcal{F}(L_1) \otimes_\infty \mathcal{F}(L_2)$.

For the applications in symplectic geometry it is important to understand the space of bounding cochains, or Maurer–Cartan elements, of a filtered A_∞ -algebra. These are solutions of the equation

$$\sum_{k \geq 0} \mathfrak{m}_k(x, \dots, x) = \mathcal{P}(x)e_A,$$

where e_A is the unit of A and $\mathcal{P}(x)$ is some element in Λ_0 . We denote by $MC(A)$ the set of solutions to this equation, modulo an equivalence relation known as gauge equivalence. Given a bounding cochain x we can deform A to obtain an A_∞ -algebra $(\hat{A}, \mathfrak{m}^x)$ with $\mathfrak{m}_0^x = \mathcal{P}(x)e_A$. This is a classical A_∞ -algebra over the Novikov field Λ . We prove the following

Theorem 1.3. *Let A and B be filtered A_∞ -algebras. There is a map*

$$\boxtimes : MC(A) \times MC(B) \rightarrow MC(A \otimes_\infty B),$$

which satisfies $\mathcal{P}(x \boxtimes y) = \mathcal{P}(x) + \mathcal{P}(y)$. When A and B are graded, connected A_∞ -algebras, this map is a bijection. Moreover there is a (strict) quasi-isomorphism

$$(\hat{A}, \mathfrak{m}^x) \otimes_\infty (\hat{B}, \mathfrak{m}^y) \simeq (\widehat{A \otimes_\infty B}, \mathfrak{m}^{\otimes, x \boxtimes y}).$$

This paper is organized in the following way. In Section 2, we review the main aspects of the theory of filtered A_∞ -algebras that we will use, mostly following [5]. In Section 3, we construct the filtered dg-algebra End_A , define the tensor product of filtered A_∞ -algebras and establish its basic properties. In Section 4, we use the homological perturbation lemma to give a model for the tensor product $A \otimes_\infty B$ on the vector space $A \otimes B$. This will allow the comparison with the previous definitions of tensor product and prove the last part of Theorem 1.1. In Section 5, we define commuting subalgebras and prove Theorem 1.2. In the last section we study bounding cochains on the tensor product $A \otimes_\infty B$ and prove Theorem 1.3.

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Conventions: Given an homogeneous element a in a graded module A , we will denote its degree by $|a|$. We will also use a shifted degree, $||a|| = |a| - 1$.

We use bold letters to denote elements in a tensor algebra,

$$\mathbf{a} = a_1 \otimes \dots \otimes a_n \in A^{\otimes n}$$

and $||\mathbf{a}|| = \sum_{i=1}^n ||a_i||$. Furthermore, we use Sweedler notation for the standard coproduct on a tensor coalgebra, namely:

$$\Delta(\mathbf{a}) = \sum_{(\mathbf{a})} \mathbf{a}^{(1)} \otimes \mathbf{a}^{(2)} = \sum_{i=0}^n (a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_n).$$

In this paper, a *tree* T is always a planar tree, that is a finite graph with no cycles embedded in the plane. We say T has k leaves when it has a distinguished set of $k + 1$ vertices with one incident edge, one of them denoted as the *root* and the others as the *leaves*. The other vertices are called internal and we denote the set of these by $V(T)$. We denote by $E(T)$ the set of edges and by $E_{int}(T)$ the set of internal edges, that is not incident at the root or a leaf. Given a vertex v there is a unique path on T from v to the root. The unique edge incident at v that is part of this path is said to be *outgoing* and the other edges incident at v are *incoming*. The *valency* of v , $val(v)$ is the number of incoming edges.

2 Filtered A_∞ -algebras

2.1 A_∞ -algebras and homomorphisms

In this section we will review the definitions and main properties of filtered A_∞ -algebras and homomorphisms introduced in [5]. We start with the definition of a general A_∞ -algebra.

Definition 2.1. An A_∞ -algebra over a ring R consists of a \mathbb{Z}_2 -graded R -module A and a collection of multilinear maps $\mathbf{m}_k : A^{\otimes k} \rightarrow A$ for each $k \geq 0$ of degree $k \pmod{2}$ satisfying the following equation

$$\sum_{\substack{0 \leq j \leq n \\ 1 \leq i \leq n-j+1}} (-1)^* \mathbf{m}_{n-j+1}(a_1, \dots, \mathbf{m}_j(a_i, \dots, a_{i+j-1}), \dots, a_n) = 0 \quad (1)$$

where $*$ = $\sum_{l=1}^{i-1} ||a_l||$.

When R is a field and $\mathfrak{m}_0 = 0$, we say A is a classical A_∞ -algebra. The A_∞ -algebra is graded if A is a \mathbb{Z} -graded module and the maps \mathfrak{m}_k have degree $2 - k$.

Remark 2.2. From now on given any expression which consists of a multilinear map applied to a block of entries, like (1) for example, we write $*$ for the sum of the shifted degrees of the entries lying to the left of the block.

In this paper we will be interested in a particular kind of A_∞ -algebra defined over the Novikov ring. Let \mathbb{K} be a field, the Novikov ring over \mathbb{K} is defined as follows

$$\Lambda_0^{\mathbb{K}} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R}, a_i \in \mathbb{K}, 0 \leq \dots \leq \lambda_i \leq \lambda_{i+1} \leq \dots, \lim_{\lambda_i \rightarrow \infty} = +\infty \right\},$$

with maximal ideal

$$\Lambda_+^{\mathbb{K}} = \left\{ \sum_i a_i T^{\lambda_i} \mid \lambda_i > 0, \forall i \text{ with } a_i \neq 0 \right\}.$$

Localizing at $\Lambda_+^{\mathbb{K}}$ we obtain the Novikov field

$$\Lambda^{\mathbb{K}} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R}, a_i \in \mathbb{K}, \lambda_i \leq \lambda_{i+1}, \lim_{\lambda_i \rightarrow \infty} = +\infty \right\}.$$

Note also that $\Lambda_0^{\mathbb{K}}$ has a natural filtration

$$F^\lambda \Lambda_0^{\mathbb{K}} = \left\{ \sum_i a_i T^{\lambda_i} \mid \lambda_i \geq \lambda, \forall i \text{ with } a_i \neq 0 \right\}.$$

Usually we will drop \mathbb{K} from the notation and simply write Λ_0 , Λ and Λ^+ .

Next, consider $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ and write $E : G \rightarrow \mathbb{R}_{\geq 0}$ and $\mu : G \rightarrow 2\mathbb{Z}$ for the natural projections. We say G is a *discrete submonoid*, if it is an additive submonoid satisfying

$$E^{-1}([0, c]) \text{ is finite for any } c \geq 0.$$

Definition 2.3. Let G be a discrete submonoid, a G -gapped filtered A_∞ -algebra $A = (A, \mathfrak{m})$ consists of a \mathbb{Z} -graded \mathbb{K} -vector space A together with maps $\mathfrak{m}_{k,\beta} : A^{\otimes k} \rightarrow A$, for each $\beta \in G$ and $k \geq 0$ of degree $2 - k - \mu(\beta)$. These are required to satisfy $\mathfrak{m}_{0,0} = 0$ and for all $\beta \in G$ and homogeneous $a_1, \dots, a_n \in A$:

$$\sum_{\substack{\beta_1 + \beta_2 = \beta \\ 0 \leq j \leq n \\ 1 \leq i \leq n-j+1}} (-1)^* \mathfrak{m}_{n-j+1, \beta_2}(a_1, \dots, \mathfrak{m}_{j, \beta_1}(a_i, \dots, a_{i+j-1}), \dots, a_n) = 0. \quad (2)$$

We will say (A, \mathfrak{m}) is a filtered A_∞ -algebra if it is G -gapped filtered for some G .

Given a filtered A_∞ -algebra (A, \mathfrak{m}) , the tensor product $A \otimes_{\mathbb{K}} \Lambda_0$ inherits from Λ_0 a filtration. We denote by $\hat{A}_0 = A \hat{\otimes} \Lambda_0$ the completion of $A \otimes_{\mathbb{K}} \Lambda_0$ with respect to this filtration and define maps $\mathfrak{m}_k : \hat{A}_0^{\otimes k} \rightarrow \hat{A}_0$ by setting

$$\mathfrak{m}_k = \sum_{\beta \in G} \mathfrak{m}_{k,\beta} T^{\omega(\beta)}.$$

The gapped condition ensures this well defined and (2) implies that $(\hat{A}_0, \mathfrak{m})$ is an A_∞ -algebra over Λ_0 .

Note that $(A, \mathfrak{m}_{k,0})$ is a classical A_∞ -algebra over \mathbb{K} . Therefore, by considering $G = \{0\}$ -gapped A_∞ -algebras we recover the theory of classical A_∞ -algebras over the field \mathbb{K} .

Next we introduce some terminology.

Definition 2.4. Let (A, \mathfrak{m}) be a G -gapped filtered A_∞ -algebra.

1. If $\mathfrak{m}_{k,\beta} = 0$ when $k > 2$ or when $k = 2$ and $\beta \neq 0$, we say A is a filtered dg-algebra.
2. If $G \subset \mathbb{R}_{\geq 0} \times \{0\}$, we say A is graded.
3. If $\mathfrak{m}_0 = 0$ we say A is a flat A_∞ -algebra.

Definition 2.5. An element $e_A \in A$ of degree 0 is said to be a *unit* of the filtered A_∞ -algebra (A, \mathfrak{m}) if

$$\mathfrak{m}_{2,0}(e_A, a) = (-1)^{|a|} \mathfrak{m}_{2,0}(a, e_A) = a$$

and $\mathfrak{m}_{k,\beta}(\dots, e, \dots) = 0$ for $(k, \beta) \neq (2, 0)$.

This notion of unit is sometimes called *strict unit*. There are other, more flexible, notions of unit, namely cohomological unit or homotopy unit. As it is explained in [14], these notions are equivalent.

Definition 2.6. Let (A, \mathfrak{m}^A) and (B, \mathfrak{m}^B) be filtered A_∞ -algebras (for some discrete submonoid G). A *filtered A_∞ -homomorphism* $F : A \longrightarrow B$ consists of a sequence of maps

$$F_{k,\beta} : A^{\otimes k} \longrightarrow B, \quad k \geq 0, \quad \beta \in G,$$

of degree $1 - k - \mu(\beta)$, satisfying

1. $F_{0,0} = 0$,
2. for each $n \geq 1$, $\beta \in G$ and $a_1, \dots, a_n \in A$

$$\begin{aligned} \sum \mathfrak{m}_{l,\beta_0}^B(F_{k_1,\beta_1}(a_1, \dots, a_{k_1}), \dots, F_{k_l,\beta_l}(\dots, a_n)) = \\ = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ 0 \leq j \leq n}} (-1)^* F_{n-j+1,\beta_2}(a_1, \dots, \mathfrak{m}_{j,\beta_1}^A(a_i, \dots, a_{i+j-1}), \dots, a_n) \end{aligned}$$

where the sum on the left hand side is taken over $l \geq 1, k_i \geq 0, \beta_i \in G$ such that $\beta_0 + \beta_1 + \dots + \beta_l = \beta$ and $k_1 + \dots + k_l = n$.

For $n = 0$ one must have

$$\mathfrak{m}_{0,\beta}^B + \sum_{l \geq 1} \mathfrak{m}_{l,\beta_0}^B(F_{0,\beta_1}, \dots, F_{0,\beta_l}) = \sum_{\beta_1 + \beta_2 = \beta} F_{1,\beta_1}(\mathfrak{m}_{0,\beta_2}^A).$$

Definition 2.7. Let $F : A \longrightarrow B$ be a filtered A_∞ -homomorphism.

1. If A and B have units e_A and e_B , we say F is *unital* if

$$F_{1,0}(e_A) = e_B \text{ and } F_{k,\beta}(\dots, e_A, \dots) = 0 \text{ for } (k, \beta) \neq (1, 0).$$

2. We say F is *strict* if $F_{0,\beta} = 0$ for all β .
3. We say F is *naive* if $F_{k,\beta} = 0$ for all $(k, \beta) \neq (1, 0)$.
4. If $F_{1,0} : A \longrightarrow B$ is an isomorphism we say F is an *isomorphism*. If the induced map on cohomology $(F_{1,0})_* : H^*(A, \mathfrak{m}_{1,0}^A) \longrightarrow H^*(B, \mathfrak{m}_{1,0}^B)$ is an isomorphism we say F is a *quasi-isomorphism*.

Given two filtered A_∞ -homomorphism $F : A \longrightarrow B$ and $G : B \longrightarrow C$ we can define their composition as

$$(G \circ F)_{n,\beta}(a_1, \dots, a_n) = \sum G_{l,\beta_0}(F_{k_1,\beta_1}(a_1, \dots, a_{k_1}), \dots, F_{k_l,\beta_l}(\dots, a_n))$$

where we sum over $l, \beta_i, k_i \geq 0$, such that $k_1 + \dots + k_l = n$ and $\beta_0 + \beta_1 + \dots + \beta_l = \beta$.

In [5], the authors develop the homotopy theory of filtered A_∞ -algebras, which we quickly review. Given a filtered A_∞ -algebra (A, \mathfrak{m}) one can define a filtered A_∞ -algebra $(A^{[0,1]}, \mathfrak{m}^{[0,1]})$ and naive filtered A_∞ -homomorphisms $i : A \longrightarrow A^{[0,1]}$, $p_0 : A^{[0,1]} \longrightarrow A$ and $p_1 : A^{[0,1]} \longrightarrow A$ satisfying:

1. i, p_0 and p_1 are quasi-isomorphism,
2. $p_1 \circ i = p_0 \circ i = id_A$,
3. $(p_1)_{1,0} \oplus (p_0)_{1,0} : A^{[0,1]} \longrightarrow A \oplus A$ is surjective.

These properties uniquely determine $A^{[0,1]}$ in an appropriate sense (see [5, Chapter 4]). In Section 6 we will give a model for $A^{[0,1]}$ using the tensor product of filtered A_∞ -algebras.

We can now give the following

Definition 2.8. Two filtered A_∞ -homomorphisms $F_0, F_1 : A \longrightarrow B$ are said to be *homotopic* if there is a filtered A_∞ -homomorphism $F^{[0,1]} : A \longrightarrow B^{[0,1]}$ such that

$$F_0 = p_0 \circ F^{[0,1]} \text{ and } F_1 = p_1 \circ F^{[0,1]}.$$

A filtered A_∞ -homomorphism $F : A \longrightarrow B$ is called a *homotopy equivalence* if there exists $G : B \longrightarrow A$ such that $F \circ G$ and $G \circ F$ are homotopic to the identity (the identity is the naive map defined by $(id)_{1,0} = id$).

The following is one of the the most important theorems about A_∞ -algebras, it is sometimes referred to as the Whitehead theorem for A_∞ -algebras.

Theorem 2.9. [5, Section 4.5] *Let $F : A \longrightarrow B$ be a filtered A_∞ -homomorphism. If F is a quasi-homomorphism then it is a homotopy equivalence. If F is unital (respectively strict) then its homotopy inverse can also be taken to be unital (respectively strict).*

2.2 Homological perturbation lemma

In this subsection we will review the homological perturbation lemma. This allows us to transfer the A_∞ -algebra structure (A, \mathfrak{m}) to a different chain complex homotopic to A .

Consider the following situation: let (A, \mathfrak{m}) be a filtered A_∞ -algebra, (V, d) be a chain complex and suppose we have chains maps $i : (V, d) \longrightarrow (A, \mathfrak{m}_{1,0})$, $p : (A, \mathfrak{m}_{1,0}) \longrightarrow (V, d)$ and a homotopy $H : A \longrightarrow A$ satisfying,

$$i \circ p - id_A = \mathfrak{m}_{1,0} \circ H + H \circ \mathfrak{m}_{1,0}. \quad (3)$$

We call such collection of maps *homotopy data*. We have the following

Theorem 2.10. *In the situation described, V has the structure of filtered A_∞ -algebra with $\eta_{1,0} = d$ and there is a filtered A_∞ -homomorphism $\varphi : (V, \eta) \longrightarrow (A, \mathfrak{m})$ with $\varphi_{1,0} = i$. Moreover there is a filtered A_∞ -homomorphism $\psi : A \longrightarrow V$ with $\psi_{1,0} = p$ and a homotopy between $\varphi \circ \psi$ and id_A .*

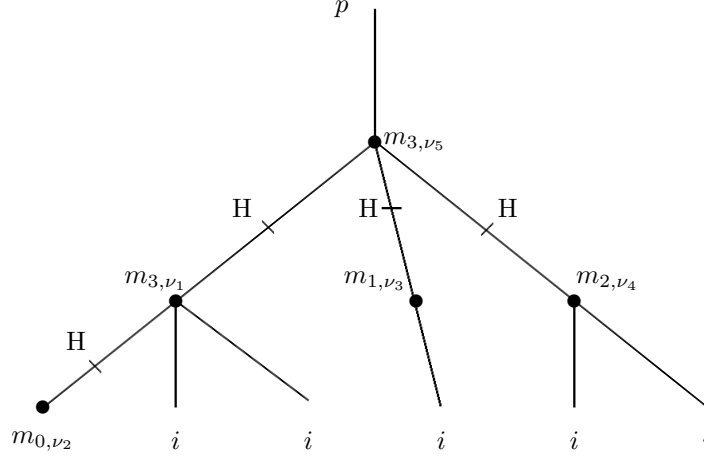


Figure 1: Example of an element of $\Gamma_5(\sum_{i=1}^5 \nu_i)$. Note that $\nu_3, \nu_2 \neq 0$.

A proof of this theorem for classical A_∞ -algebras, together with explicit formulas for the maps is given in [9]. For the case of filtered A_∞ -algebras see [4]. Here we will only give a description of the maps $\eta_{k,\beta}$ and $\varphi_{k,\beta}$.

Denote by $\Gamma_k(G)$ the set of trees with k -leaves and a map $\nu : V(T) \rightarrow G$. We impose the condition that $\nu(v) \neq 0$ if v is an internal vertex of valency zero or one.

Now, given $\beta \in G$, we consider $\Gamma_k(\beta)$ the subset of $\Gamma_k(G)$ of trees T such that $\sum_{v \in V(T)} \nu(v) = \beta$. We can easily see that $\Gamma_k(\beta)$ is finite. For each $T \in \Gamma_k(\beta)$, we define \bar{T} as the tree T with one additional vertex in each internal edge of T . We use T as a flow chart to define a map

$$\eta_T : V^{\otimes k} \rightarrow V. \quad (4)$$

We assign to each $v \in V(T)$ the map $\mathbf{m}_{val(v), \nu(v)}$; to the vertices in \bar{T} but not T we assign H ; and finally we assign p to the root and i to the leaves. For example, the tree in Figure 1 gives the map

$$\begin{aligned} \eta_T(v_1, v_2, v_3, v_4, v_5) = \\ = p(\mathbf{m}_{3,\nu_5}(H \circ \mathbf{m}_{3,\nu_1}(H(\mathbf{m}_{0,\nu_2}), i(v_1), i(v_2))), H \circ \mathbf{m}_{1,\nu_3}(i(v_3)), H \circ \mathbf{m}_{2,\nu_4}(i(v_4), i(v_5))). \end{aligned}$$

Then we define

$$\eta_{k,\beta} = \sum_{T \in \Gamma_k(\beta)} \eta_T.$$

The construction of the map $\varphi : V \rightarrow A$ is similar. We take $\varphi_{k,\beta} = \sum_{T \in \Gamma_k(\beta)} \varphi_T$. The map φ_T is defined in the same way as η_T , the only difference is that we assign H to the root vertex (instead of p as in the case of η_T).

When A is unital, we can ensure V and φ are also unital by imposing extra conditions on p, i and H . These are known as side conditions.

Proposition 2.11. *Suppose (A, \mathbf{m}) is unital and that the homotopy data satisfies*

$$p \circ i = id, \quad H \circ i = 0, \quad p \circ H = 0 \quad \text{and} \quad H^2 = 0.$$

Additionally assume there is e_V such that $i(e_V) = e_A$. Then e_V is a unit for (V, η) and φ is a unital quasi-isomorphism.

PROOF. Simply inspecting the formulas given above for $\eta_{k,\beta}$ and applying the side conditions we see that

$$\eta_{k+1,\beta}(v_1, \dots, e_V, \dots, v_k) = p(\mathfrak{m}_{k,\beta}(i(v_1), \dots, e_A, \dots, i(v_k))),$$

which readily implies e_V is a unit. The same argument shows φ is unital. The first condition $p \circ i = id$ together with the homotopy imply that $i = \varphi_{1,0}$ is an isomorphism in cohomology. Thus, we conclude φ is a unital quasi-isomorphism. \square

The standard application of the homological perturbation lemma is the following proposition which is proved in [4].

Proposition 2.12. *Any filtered A_∞ -algebra (A, \mathfrak{m}) is quasi-isomorphic to a filtered A_∞ -algebra (A', \mathfrak{m}') with $\mathfrak{m}'_{1,0} = 0$. We call (A', \mathfrak{m}') a canonical model for A .*

PROOF. We pick a subspace $W \subseteq A$ such that $W \oplus \text{Ker } \mathfrak{m}_{1,0} = A$. Since $\mathfrak{m}_{1,0}^2 = 0$ we can choose $H \subseteq \text{Ker } \mathfrak{m}_{1,0}$ satisfying $H \oplus \mathfrak{m}_{1,0}(W) = \text{Ker } \mathfrak{m}_{1,0}$. This gives a decomposition

$$A = H \oplus W \oplus \mathfrak{m}_{1,0}(W).$$

Note that if H is nontrivial we can choose it so that $e_A \in H$. Using the decomposition we define an inclusion $i : H \rightarrow A$, a projection $p : A \rightarrow H$ and a map $H : A \rightarrow A$ of degree -1 which is zero when restricted to $H \oplus W$ and is the inverse of $\mathfrak{m}_{1,0}$ on $\mathfrak{m}_{1,0}(W)$. It's straightforward to check that this defines homotopy data satisfying the side conditions. Applying the homological perturbation lemma to this data we obtain a filtered A_∞ -algebra $(H, \eta_{k,\beta})$ with $\eta_{1,0} = 0$ quasi-isomorphic to A . \square

2.3 Bounding cochains

Here we will review some basic definitions and facts about bounding cochains, see [5, Section 4.3] for further details.

Definition 2.13. Let (A, \mathfrak{m}) be a $(G$ -gapped) filtered A_∞ -algebra and consider $x = \sum_{i=0}^{\infty} x_i T^{\lambda_i} \in \hat{A}_0$ with $\lambda_i \geq \lambda$ for some $\lambda > 0$ and x_i of odd degree. We say x is a *bounding cochain* if there is $\mathcal{P}(x) \in \Lambda_0$ such that

$$\sum_{k \geq 0} \mathfrak{m}_k(x, \dots, x) = \mathcal{P}(x)e_A. \quad (5)$$

Note that in [5], the authors use the term *weak bounding cochain* and reserve the term *bounding cochain* to the case when $\mathcal{P}(x) = 0$. Bounding cochains are sometimes also called Maurer–Cartan elements. We denote by $\bar{MC}(A)$ the set of all bounding cochains.

In the case A is graded, we require that all the x_i have degree one. Observe that in this case $\mathcal{P}(x) = 0$, since the left hand side of (5) has degree 2 and e_A has degree 0.

Given a bounding cochain x we deform the A_∞ -algebra A in the following way. Consider $\gamma_i = (\lambda_i, 1 - |x_i|) \in \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ and let \bar{G} be the smallest submonoid containing G and all the γ_i . This is a discrete submonoid. We then define a new \bar{G} -gapped A_∞ -algebra (A, \mathfrak{m}^x) , by setting

$$\mathfrak{m}_k^x(a_1, \dots, a_k) = \sum_{i_0, \dots, i_k} \mathfrak{m}_{k+i_0+\dots+i_k}(x, \dots, x, a_1, x, \dots, x, a_k, x, \dots, x).$$

Observe that, if A is graded so is (A, \mathfrak{m}^x) .

Also note that $\mathfrak{m}_0^b = \mathcal{P}(x)e_A$ therefore, by the A_∞ -relation, $\mathfrak{m}_1^x : \hat{A}_0 \rightarrow \hat{A}_0$ squares to zero. Thus given a bounding cochain x we can define the cohomology for the pair (A, x) :

$$H^*(A, x; \Lambda_0) := \frac{\text{Ker } \mathfrak{m}_1^x}{\text{Im } \mathfrak{m}_1^x}.$$

Often, it is more convenient to work over a field so we consider the A_∞ -algebra $(\hat{A}, \mathfrak{m}^x)$ where $\hat{A} = \hat{A}_0 \otimes_{\Lambda_0} \Lambda$. Observe that this is a classical A_∞ -algebra over the field Λ , even when $\mathcal{P}(x) \neq 0$. This is because $\mathfrak{m}_0^x = \mathcal{P}(b)e_A$ and therefore the terms in the A_∞ -equations involving \mathfrak{m}_0^x cancel. Hence we can think of $(\hat{A}, \mathfrak{m}^x)$ as a classical A_∞ -algebra with one additional invariant $\mathcal{P}(x) \in \Lambda_0$.

Next we introduce an equivalence relation on $\widehat{MC}(A)$. We say $x_0, x_1 \in \widehat{MC}(A)$ are *gauge equivalent*, and we write $x_1 \sim x_0$, if there is $\bar{x} \in \widehat{MC}(A^{[0,1]})$ such that $p_0(\bar{x}) = x_0$ and $p_1(\bar{x}) = x_1$. We denote by $MC(A)$ the set of equivalence classes. It is proved in [5, Section 4.3] that if $x_1 \sim x_0$, then $\mathcal{P}(x_1) = \mathcal{P}(x_0)$, so \mathcal{P} is a well defined function on $MC(A)$. Moreover $H^*(A, x_0; \Lambda_0) \cong H^*(A, x_1; \Lambda_0)$.

The following theorem is proved in [5] in Sections 4.3 and 5.2.

Theorem 2.14. *Let $F : (A, \mathfrak{m}^A) \longrightarrow (B, \mathfrak{m}^B)$ be a unital quasi-isomorphism. This induces a bijection $F_* : MC(A) \longrightarrow MC(B)$. Moreover $\mathcal{P}(F_*(x)) = \mathcal{P}(x)$,*

$$H^*(A, x; \Lambda_0) \cong H^*(B, F_*(x); \Lambda_0)$$

and $(\hat{A}, \mathfrak{m}^b)$ is quasi-isomorphic to $(\hat{B}, \mathfrak{m}^{F_*(b)})$, as classical A_∞ -algebras over Λ .

3 Tensor product of filtered A_∞ -algebras

3.1 Definition of the tensor product

We will now consider the problem of defining the tensor product of filtered A_∞ -algebras. We will first prove that any A_∞ -algebra is quasi-isomorphic to a filtered dg-algebra and then define the tensor product for these. This approach is inspired on the definition of tensor product of classical A_∞ -algebras used in [7].

Let (A, \mathfrak{m}) be a unital filtered A_∞ -algebra. We define End_A to be the subspace of

$$\text{Hom}_{\mathbb{K}} \left(\bigoplus_{s \geq 0} A \otimes A^{\otimes s}, A \right)$$

of elements satisfying $\rho_s(\bullet, \dots, e_A, \dots) = 0$. An element $\rho = \{\rho_s\}_s \in End_A$ has degree $|\rho|$ if each ρ_s has degree $|\rho| - s$. For each $\beta \in G$ we define the operations

$$(\mu_{0,\beta})_s(v, a_1, \dots, a_s) = \sum_{\beta_1 + \beta_2 = \beta} \mathfrak{m}_{s+2, \beta_1}(\mathfrak{m}_{0, \beta_2}, v, a_1, \dots, a_s),$$

$$\begin{aligned} (\mu_{1,\beta}(\rho))_s(v, a_1, \dots, a_s) &= \sum_{i \geq 0} -\mathfrak{m}_{s-i+1, \beta}(\rho_i(v, a_1, \dots, a_i), \dots, a_s) \\ &\quad + \sum_{i \geq 0} (-1)^{|\rho|} \rho_{s-i}(\mathfrak{m}_{i+1, \beta}(v, a_1, \dots, a_i), \dots, a_s) \\ &\quad + \sum_{j, i \geq 0} (-1)^{|\rho| + *} \rho_{s-j+1}(v, \dots, \mathfrak{m}_{j, \beta}(a_{i+1}, \dots, a_{i+j}), \dots, a_s), \end{aligned}$$

$$(\mu_{2,0}(\rho, \tau)) = (-1)^{|\rho|} \rho \circ \tau$$

$$\text{where } (\rho \circ \tau)_s(v, a_1, \dots, a_s) = \sum_{i \geq 0} \rho_{s-i}(\tau_i(v, a_1, \dots, a_i), \dots, a_s).$$

For all the other (k, β) we make $\mu_{k, \beta} = 0$.

When A is a flat A_∞ -algebra, one can regard A as a (right) A_∞ -module over itself. In this case, End_A is simply the filtered A_∞ -algebra of A_∞ -pre-homomorphisms of this module (see [13] for this). In general, A is not a right module over itself, but as we will see End_A is still a filtered A_∞ -algebra.

Lemma 3.1. *$End_A = (End_A, \mu)$ is an unital filtered dg-algebra.*

PROOF. The proof consists of a series of long and tedious computations. We start with

$$\begin{aligned} & \left[\sum_{\beta_1 + \beta_2 = \beta} \mu_{1, \beta_1}(\mu_{0, \beta_2}) \right]_s (v, a_1, \dots, a_s) = \\ &= \sum_{\beta_1 + \beta' + \beta'' = \beta} -\mathbf{m}_{s-i+1, \beta_1}(m_{i+2, \beta'}(\mathbf{m}_{0, \beta''}, v, \dots, a_i), \dots, a_s) \\ &+ \sum_{\substack{i \geq 0 \\ \beta_1 + \beta' + \beta'' = \beta}} \mathbf{m}_{s-i+2, \beta'}(\mathbf{m}_{0, \beta''}, \mathbf{m}_{i+1, \beta_1}(v, \dots, a_i), \dots, a_s) \\ &+ \sum_{\substack{i, j \geq 0 \\ \beta_1 + \beta' + \beta'' = \beta}} (-1)^{*+1} \mathbf{m}_{s+j+1, \beta'}(\mathbf{m}_{0, \beta''}, v, \dots, \mathbf{m}_{j, \beta_1}(a_{i+1}, \dots, a_{i+j}), \dots, a_s). \end{aligned}$$

By the A_∞ -equation this equals

$$\sum_{\beta''} \sum_{\beta_1 + \beta' = \beta - \beta''} \mathbf{m}_{s+3, \beta'}(\mathbf{m}_{0, \beta_1}, \mathbf{m}_{0, \beta''}, v, \dots, a_s) - \mathbf{m}_{s+3, \beta'}(\mathbf{m}_{0, \beta''}, \mathbf{m}_{0, \beta_1}, v, \dots, a_s) = 0.$$

So we conclude $\sum_{\beta_1 + \beta_2 = \beta} \mu_{1, \beta_1}(\mu_{0, \beta_2}) = 0$. The next equation can be handled in the same way:

$$\begin{aligned} \sum_{\beta_1 + \beta_2 = \beta} \mu_{1, \beta_1}(\mu_{1, \beta_2}(\rho))_s(v, a_1, \dots, a_s) &= \sum_{i \geq 0} -\mathbf{m}_{s-i+1, \beta_1}((\mu_{1, \beta_2}(\rho))_i(v, \dots, a_i), \dots, a_s) + \\ &+ \sum_{i \geq 0} (-1)^{|\rho|+1} (\mu_{1, \beta_2}(\rho))_{s-i}(\mathbf{m}_{i+1, \beta_1}(v, \dots, a_i), \dots, a_s) + \\ &+ \sum_{i, j \geq 0} (-1)^{|\rho|+*} (\mu_{1, \beta_2}(\rho))_s(v, \dots, \mathbf{m}_{j, \beta_1}(a_{i+1}, \dots, a_{i+j}), \dots, a_s). \end{aligned}$$

Further expanding and canceling all the terms we see this is equal to

$$\begin{aligned} & \sum_{j \geq 0} \sum_{\substack{\beta_1 + \beta_2 = \beta \\ i \geq j}} \mathbf{m}_{s-i+1, \beta_1}(\mathbf{m}_{i-j+1, \beta_2}(\rho_j(v, \dots, a_j), \dots, a_i) \dots a_s) \\ &+ \sum_{j \geq 0} \sum_{\substack{\beta_1 + \beta_2 = \beta \\ l \geq j \\ i \geq 0}} (-1)^{|\rho|+|v|+\sum_{p=1}^l \|a_p\|} \mathbf{m}_{s-i-j+2, \beta_1}(\rho_j(v, \dots, a_j), \dots, \mathbf{m}_{i, \beta_2}(a_{l+1}, \dots, a_{l+i}), \dots, a_s) \\ &- \sum_j \sum_{\substack{\beta_1 + \beta_2 = \beta \\ 0 \leq i \leq j}} \rho_{s-j}(\mathbf{m}_{j-i+1, \beta_1}(\mathbf{m}_{i+1, \beta_2}(v, \dots, a_i) \dots a_j), \dots, a_s) \\ &- \sum_j \sum_{\substack{\beta_1 + \beta_2 = \beta \\ l \geq 0 \\ i \geq 0}} (-1)^* \rho_{s-j}(\mathbf{m}_{j-i+2, \beta_1}(v, \dots, \mathbf{m}_{i, \beta_2}(a_{l+1}, \dots, a_{l+i}), \dots, a_j), \dots, a_s). \end{aligned}$$

Adding the first two terms and the last two, and using the A_∞ -equation, we obtain

$$\begin{aligned}
& - \sum_{\beta_1 + \beta_2 = \beta} \sum_j \mathbf{m}_{s-j+2, \beta_1}(\mathbf{m}_{0, \beta_2}, \rho(v, \dots, a_j), \dots, a_s) + \\
& \quad + \sum_{\beta_1 + \beta_2 = \beta} \sum_j \rho_{s-j}(\mathbf{m}_{j+2, \beta_1}(\mathbf{m}_{0, \beta_2}, v, \dots, a_j), \dots, a_s) = \\
& = \left[-\mu_{2,0}(\mu_{0,\beta}, \rho) + (-1)^{|\rho|} \mu_{2,0}(\rho, \mu_{0,\beta}) \right]_s (v, a_1, \dots, a_s).
\end{aligned}$$

So we conclude that

$$\sum_{\beta_1 + \beta_2 = \beta} \mu_{1, \beta_1}(\mu_{1, \beta_2}(\rho)) + \mu_{2,0}(\mu_{0,\beta}, \rho) + (-1)^{||\rho||} \mu_{2,0}(\rho, \mu_{0,\beta}) = 0.$$

The last two equations we have to check

$$\mu_{2,0}(\mu_{1,\beta}(\rho), \tau) + (-1)^{||\rho||} \mu_{2,0}(\rho, \mu_{1,\beta}(\tau)) + \mu_{1,\beta}(\mu_{2,0}(\rho, \tau)) = 0,$$

and

$$\mu_{2,0}(\mu_{2,0}(\rho, \tau), \theta) + (-1)^{||\rho||} \mu_{2,0}(\rho, \mu_{2,0}(\tau, \theta)) = 0,$$

simply state that (up to a change in sign) $\mu_{2,0}$ is associative and $\mu_{1,\beta}$ is a derivation of $\mu_{2,0}$. They do not involve $\mu_{0,\beta}$ and so they can be deduced from the flat case, which is done in [13].

Finally we can easily verify that

$$(id)_r(v, a_1, \dots, a_r) = \begin{cases} v & , r = 0 \\ 0 & , \text{otherwise} \end{cases}$$

defines a unit in End_A .

□

Now, we will show that End_A is quasi-isomorphic to A . We start with the following

Proposition 3.2. *Let A be a unital filtered A_∞ -algebra and End_A be the filtered dg-algebra described above. Consider $F_{k,\beta} : A^{\otimes k} \rightarrow End_A$ defined as*

$$F_{k,\beta}(\alpha_1, \dots, \alpha_k)_s(v, a_1, \dots, a_s) = \mathbf{m}_{k+s+1, \beta}(\alpha_1, \dots, \alpha_k, v, a_1, \dots, a_s),$$

for $k > 0$. This defines a strict A_∞ -homomorphism.

PROOF. First note that

$$\mu_{0,\beta} = \sum_{\beta_1 + \beta_2 = \beta} F_{1, \beta_1}(\mathbf{m}_{0, \beta_2})$$

by definition. Next we need to check

$$\begin{aligned}
& \sum_{\substack{\beta_1 + \beta_2 = \beta \\ 1 \leq i \leq k-1}} (-1)^{|F_i(\alpha_1, \dots, \alpha_i)|} F_{i, \beta_1}(\alpha_1, \dots, \alpha_i) \circ F_{k-i, \beta_2}(\alpha_{i+1}, \dots, \alpha_k) + \mu_{1, \beta_1}(F_{k, \beta_2}(\alpha_1, \dots, \alpha_k)) \\
& = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ 0 \leq j \leq k}} (-1)^* F_{k-j+1, \beta_1}(\alpha_1, \dots, \mathbf{m}_{j, \beta_2}(\alpha_{i+1}, \dots, \alpha_{i+j}), \dots, \alpha_k).
\end{aligned}$$

If we denote by L and R the left and right-hand side of the above equation, we need to check $L_s(v, a_1, \dots, a_s) = R_s(v, a_1, \dots, a_s)$. Noting that $|F_i(\alpha_1, \dots, \alpha_i)| = \sum_{l=1}^i \|\alpha_l\| + 1$, we compute

$$\begin{aligned} L_s(v, a_1, \dots, a_s) &= \sum_{0 \leq j \leq s} -\mathbf{m}_{s-j+1, \beta_1}(\mathbf{m}_{k+j+1, \beta_2}(\alpha_1, \dots, \alpha_k, v, a_1, \dots, a_j), \dots, a_s) \\ &+ \sum_{0 \leq j \leq s} (-1)^{*+1} \mathbf{m}_{k+s-j, \beta_2}(\alpha_1, \dots, \alpha_k, \mathbf{m}_{j+1, \beta_1}(v, a_1, \dots, a_j), \dots, a_s) \\ &+ \sum_{\substack{0 \leq j \leq s \\ 0 \leq l \leq s-j}} (-1)^{*+1} \mathbf{m}_{k+s-j, \beta_2}(\alpha_1, \dots, \alpha_k, v, \dots, \mathbf{m}_{j, \beta_1}(a_{l+1}, \dots, a_{l+j}), \dots, a_s) \\ &+ \sum_{\substack{0 \leq j \leq s \\ 1 \leq i \leq k-1}} (-1)^{*+1} \mathbf{m}_{i+s-j+1, \beta_1}(\alpha_1, \dots, \alpha_i, \mathbf{m}_{k-i+j+1, \beta_2}(\alpha_{i+1}, \dots, v, \dots, a_j), \dots, a_s) \end{aligned}$$

and

$$R_s(v, a_1, \dots, a_s) = \sum_{\substack{0 \leq j \leq k \\ 0 \leq i \leq k-j}} (-1)^* \mathbf{m}_{s+k-j, \beta_1}(\alpha_1, \dots, \mathbf{m}_{j, \beta_2}(\alpha_{i+1}, \dots, \alpha_{i+j}), \dots, v, \dots, a_s).$$

Thus we conclude that $L_s(v, a_1, \dots, a_s) - R_s(v, a_1, \dots, a_s) = 0$, since this is simply the A_∞ -equation on A . \square

Now we will see that $F_{1,0}$ induces an isomorphism on cohomology, by defining a homotopy inverse. Consider $P : \text{End}_A \rightarrow A$ defined as $P(\rho) = (-1)^{|\rho|} \rho_0(e_A)$. We can easily see that $P : (\text{End}_A, \mu_{1,0}) \rightarrow (A, \mathbf{m}_{1,0})$ is a chain map and $P \circ F_{1,0} = \text{id}_A$.

Lemma 3.3. *Define $H : \text{End}_A \rightarrow \text{End}_A$ as*

$$(H\rho)_s(v, a_1, \dots, a_s) = (-1)^{|\rho|} \rho_{s+1}(e_A, v, a_1, \dots, a_s),$$

then $F_{1,0} \circ P - \text{id}_{\text{End}_A} = \mu_{1,0} \circ H + H \circ \mu_{1,0}$. Therefore $F_{1,0}$ induces an isomorphism in cohomology.

PROOF. We compute

$$\begin{aligned} (\mu_{1,0} \circ H(\rho))_s(v, a_1, \dots, a_s) &= \sum_{0 \leq i \leq s} (-1)^{|\rho|+1} \mathbf{m}_{s-i+1,0}(\rho_{i+1}(e, v, a_1, \dots, a_i), \dots, a_s) \\ &+ \sum_{0 \leq i \leq s} \rho_{s-i}(e, \mathbf{m}_{i+1,0}(v, a_1, \dots, a_i), \dots, a_s) \\ &+ \sum_{\substack{1 \leq j \leq s \\ 0 \leq i \leq s-j}} (-1)^{*+1} \rho_{s-j}(e, v, a_1, \dots, \mathbf{m}_{j,0}(a_{i+1}, \dots, a_{i+j}), \dots, a_s). \end{aligned}$$

Similarly

$$\begin{aligned} (H \circ \mu_{1,0}(\rho))_s(v, a_1, \dots, a_s) &= \sum_{0 \leq i \leq s} (-1)^{|\rho|} \mathbf{m}_{s-i+1,0}(\rho_{i+1}(e, v, a_1, \dots, a_i), \dots, a_s) \\ &+ (-1)^{|\rho|} \mathbf{m}_{s+2,0}(\rho_0(e), v, \dots, a_s) \\ &+ \sum_{0 \leq i \leq s} -\rho_{s-i}(\mathbf{m}_{i+2,0}(e, v, a_1, \dots, a_i), \dots, a_s) \\ &+ \sum_{\substack{1 \leq j \leq s \\ 0 \leq i \leq s-j}} (-1)^* \rho_{s-j}(e, v, \dots, \mathbf{m}_{j,0}(a_{i+1}, \dots, a_{i+j}), \dots, a_s) \\ &+ \sum_{0 \leq j \leq s} -\rho_{s-j+1}(e, \mathbf{m}_{j+1,0}(v, a_1, \dots, a_j), \dots, a_s). \end{aligned}$$

Comparing the two expressions we obtain

$$\begin{aligned}
& ((\mu_{1,0}H + H\mu_{1,0})\rho)_s(v, a_1, \dots, a_s) = \\
& = (-1)^{|\rho|} \mathbf{m}_{s+2,0}(\rho_0(e), v, \dots, a_s) - \sum_{i \geq 0} \rho_{s-i}(\mathbf{m}_{i+2,0}(e, v, \dots, a_i), \dots, a_s) \\
& = (-1)^{|\rho|} \mathbf{m}_{s+2,0}(\rho_0(e), v, \dots, a_s) - \rho_s(v, a_1, \dots, a_s) \\
& = \left[((F_{1,0} \circ P)\rho) - \rho \right]_s(v, a_1, \dots, a_s). \tag{6}
\end{aligned}$$

The second equality holds because A is unital, so $\mathbf{m}_{i+2,0}(e, v, a_1, \dots, a_i) = 0$ unless $i = 0$.

Thus P is a homotopy inverse of $F_{1,0}$ which implies the result. \square

Corollary 3.4. *Let A be a unital filtered A_∞ -algebra. Then A is quasi-isomorphic to the filtered dg-algebra End_A .*

We are now ready to define the tensor product of filtered A_∞ -algebras. Given A and B filtered G_A and G_B -gapped A_∞ -algebras, we will define their tensor product as a G -gapped A_∞ -algebra, for $G = G_A + G_B$. In fact the tensor product is bi-gapped in the following sense: for each $\beta_1 \in G_A$ and $\beta_2 \in G_B$ we will define $\mathbf{m}_{k, \beta_1 \times \beta_2}^\otimes$ and given $\beta \in G_A + G_B$ we take

$$\mathbf{m}_{k, \beta}^\otimes = \sum_{\substack{\beta_1 \times \beta_2 \in G_A \times G_B \\ \beta_1 + \beta_2 = \beta}} \mathbf{m}_{k, \beta_1 \times \beta_2}^\otimes.$$

We start by defining the tensor product of filtered dg-algebras.

Lemma 3.5. *Let (C, μ^C) and (D, μ^D) be unital filtered dg-algebras. On the vector space $C \otimes_{\mathbb{K}} D$ consider the operations*

$$\mu_{0, \beta_1 \times \beta_2}^\otimes = \begin{cases} \mu_{0, \beta_1}^C \otimes e_D & , \beta_2 = 0 \\ e_C \otimes \mu_{0, \beta_2}^D & , \beta_1 = 0 \\ 0 & , \beta_1, \beta_2 \neq 0 \end{cases},$$

$$\mu_{1, \beta_1 \times \beta_2}^\otimes(c \otimes d) = \begin{cases} \mu_{1, \beta_1}^C(c) \otimes d & , \beta_2 = 0 \\ (-1)^{|c|} c \otimes \mu_{1, \beta_2}^D(d) & , \beta_1 = 0 \\ 0 & , \beta_1, \beta_2 \neq 0 \end{cases},$$

$$\mu_{2,0}^\otimes(c_1 \otimes d_1, c_2 \otimes d_2) = (-1)^{|c_2||d_1|} \mu_{2,0}^C(c_1, c_2) \otimes \mu_{2,0}^D(d_1, d_2).$$

Then $(C \otimes D, \mu^\otimes)$ is an unital filtered dg-algebra, which we denote by $C \otimes_{dg} D$

PROOF. The proof of the A_∞ equations is a simple check we omit. It is also clear that $e_A \otimes e_B$ is a unit. \square

Remark 3.6. *To avoid confusion, we spell out the notation above when $\beta = 0$, we have*

$$\mu_{1,0}^\otimes(c \otimes d) = \mu_{1,0}^C(c) \otimes d + (-1)^{|c|} c \otimes \mu_{1,0}^D(d).$$

We are now ready to state our main

Definition 3.7. Let A and B be filtered A_∞ -algebras. We define their tensor product as

$$A \otimes_\infty B := \text{End}_A \otimes_{dg} \text{End}_B.$$

If A and B are classical A_∞ -algebras over \mathbb{K} , then they are also $G = \{0\}$ -gapped A_∞ -algebras, then the above also gives a definition of classical A_∞ -algebras. For these algebras the tensor product was already defined, see [8, 10, 12]. All these constructions define the tensor product as an A_∞ -algebra structure on the vector space $A \otimes B$. Our definition uses a different underlying vector space and therefore is not directly comparable. In the next section, we will remedy this by using the homological perturbation lemma to transfer our A_∞ -structure on $A \otimes_\infty B$ to the vector space $A \otimes B$.

3.2 Some properties of the tensor product

Let A_1, A_2, B_1 and B_2 be filtered A_∞ -algebras and let $\varphi : A_1 \rightarrow A_2$ and $\psi : B_1 \rightarrow B_2$ be filtered A_∞ -homomorphisms. We will show that these induce a filtered A_∞ -homomorphism

$$\varphi \otimes_\infty \psi : A_1 \otimes_\infty B_1 \rightarrow A_2 \otimes_\infty B_2.$$

Although not functorial, this construction satisfies the additional property that if φ and ψ are quasi-isomorphisms then $\varphi \otimes_\infty \psi$ is also a quasi-isomorphism. This will be enough to show that the tensor product is well defined and satisfies some monoidal properties.

Following our strategy we start by dealing with the case of filtered dg-algebras. In the following, we will use the notation: given $c_1, \dots, c_k \in C$ we denote by $c_1 \cdots c_k = \mu_{2,0}(\dots \mu_{2,0}(\mu_{2,0}(c_1, c_2), c_3) \dots, c_k)$ with the convention that if $k = 0$ we mean e_C .

Proposition 3.8. *Let C_1, C_2, D_1 and D_2 be filtered dg-algebras. Suppose we have unital filtered A_∞ -homomorphisms $\varphi, f : C_1 \rightarrow C_2$ and $\psi, g : D_1 \rightarrow D_2$, with f and g naive. Then there are A_∞ -homomorphisms $\varphi \otimes_{dg} g, f \otimes_{dg} \psi : C_1 \otimes_{dg} D_1 \rightarrow C_2 \otimes_{dg} D_2$ defined as follows.*

$$(\varphi \otimes_{dg} g)_{k, \beta_1} = (-1)^{\triangleleft} \varphi_{k, \beta_1}(c_1, \dots, c_k) \otimes g(d_1 \cdots d_k),$$

if $\beta_1 \in G_C$ and zero otherwise.

$$(f \otimes_{dg} \psi)_{k, \beta_2} = (-1)^{\triangleleft + (k+1)|c|} f(c_1 \cdots c_k) \otimes \psi_{k, \beta_2}(d_1, \dots, d_k)$$

if $\beta_2 \in G_D$ and zero otherwise.

In both cases the sign is defined as $\triangleleft = \sum_{p > q} |c_p| |d_q|$.

PROOF. The proof of both statements is essentially the same, so we will check the A_∞ -homomorphism equation just for $\varphi \otimes_{dg} g$.

We start by observing that the A_∞ equations on a filtered dg-algebra imply the following identities:

$$\begin{aligned} (c_1 \cdots c_j)(c_{j+1} \cdots c_k) &= (-1)^{(k-j-1) \sum_{l=1}^j |c_l|} c_1 \cdots c_k, \\ c_1 \cdots (c_i c_{i+1}) \cdots c_k &= (-1)^{\sum_{l=1}^{i-1} |c_l|} c_1 \cdots c_k, \\ c_1 \cdots c_i e_C c_{i+1} \cdots c_k &= (-1)^{\sum_{l=1}^i |c_l|} c_1 \cdots c_k, \\ (-1)^k \mu_{1, \beta}(c_1 \cdots c_k) + \sum_{1 \leq i \leq k} (-1)^{\sum_{l=1}^{i-1} |c_l|} c_1 \cdots \mu_{1, \beta}(c_i) \cdots c_k &= 0. \end{aligned} \tag{7}$$

We skip the proof of the A_∞ -homomorphism equation with no inputs as it is very similar

to the general case. Thus we will prove, for each $k > 0$, that the following equation holds.

$$\begin{aligned}
& \sum_{\substack{0 \leq j \leq k \\ \beta' + \beta'' = \beta}} \mu_{2,0}^{\otimes}((\varphi \otimes g)_{j,\beta'}(c_1 \otimes d_1, \dots, c_j \otimes d_j), (\varphi \otimes g)_{k-j,\beta''}(c_{j+1} \otimes d_{j+1}, \dots, c_k \otimes d_k)) + \\
& + \sum_{\beta' + \beta'' = \beta} \mu_{1,\beta''}^{\otimes}((\varphi \otimes g)_{k,\beta'}(c_1 \otimes d_1, \dots, c_k \otimes d_k)) = \\
& = \sum_{1 \leq i \leq k-1} (-1)^*(\varphi \otimes g)_{k-1,\beta}(c_1 \otimes d_1, \dots, \mu_{2,0}^{\otimes}(c_i \otimes d_i, c_{i+1} \otimes d_{i+1}), \dots, c_k \otimes d_k) \\
& + \sum_{\substack{1 \leq i \leq k \\ \beta' + \beta'' = \beta}} (-1)^*(\varphi \otimes g)_{k,\beta'}(c_1 \otimes d_1, \dots, \mu_{1,\beta''}^{\otimes}(c_i \otimes d_i), \dots, c_k \otimes d_k) \\
& + \sum_{\substack{0 \leq i \leq k \\ \beta' + \beta'' = \beta}} (-1)^*(\varphi \otimes g)_{k+1,\beta'}(c_1 \otimes d_1, \dots, c_i \otimes d_i, \mu_{0,\beta''}^{\otimes}, c_{i+1} \otimes d_{i+1}, \dots, c_k \otimes d_k),
\end{aligned}$$

We will do it by breaking it into a sum of two equalities. The first consists of the terms where both $\beta', \beta'' \in G_C$ and of these, only those involving $\mu_{k,\beta' \times 0}^{\otimes}$. The second equation has the remaining terms, namely those involving terms of the form $\mu_{k,0 \times \beta''}^{\otimes}$. The first one equals

$$\begin{aligned}
& \sum_{\beta', \beta'' \in G_C} (-1)^{\triangleleft + (k-j-1) \sum_{l=1}^j |d_l|} \mu_{2,0}(\varphi_{j,\beta'}(c_1, \dots, c_j), \varphi_{k-j,\beta''}(c_{j+1}, \dots, c_k)) \otimes \\
& g(d_1 \cdots d_j) g(d_{j+1} \cdots d_k) + (-1)^{\triangleleft} \sum_{\beta', \beta'' \in G_C} \mu_{1,\beta'}(\varphi_{k,\beta''}(c_1, \dots, c_k)) \otimes g(d_1 \cdots d_k) \\
& = \sum_{1 \leq i \leq k-1} (-1)^{\triangleleft + \sum_{l=1}^{i-1} (|c_l| + |d_l|)} \varphi_{k-1,\beta}(c_1, \dots, \mu_{2,0}(c_i, c_{i+1}), \dots, c_k) \otimes g(d_1 \cdots (d_i d_{i+1}) \cdots d_k) \\
& + \sum_{\substack{1 \leq i \leq k \\ \beta', \beta'' \in G_C}} (-1)^{\triangleleft + \sum_{l=1}^{i-1} |c_l|} \varphi_{k,\beta'}(c_1, \dots, \mu_{1,\beta''}(c_i), \dots, c_k) \otimes g(d_1 \cdots d_k) \\
& + \sum_{\substack{\beta', \beta'' \in G_C \\ 0 \leq i \leq k}} (-1)^{\triangleleft + \sum_{l=1}^{i-1} (|c_l| + |d_l|)} \varphi_{k+1,\beta'}(c_1, \dots, \mu_{0,\beta''}, \dots, c_k) \otimes g(d_1 \cdots d_i \cdot e_{D_1} d_{i+1} \cdots d_k).
\end{aligned}$$

Using the first three identities in (7) and the fact that g is a naive A_∞ -homomorphism we conclude that this equation holds, since it is equivalent to the A_∞ -homomorphism equation for $\varphi_{k,\beta}$.

As for the second equation, we have

$$\begin{aligned}
& \sum_{\beta' + \beta'' = \beta} (-1)^{\triangleleft + k + 1 + |c|} \varphi_{k,\beta'}(c_1, \dots, c_k) \otimes \mu_{1,\beta''} g((d_1 \cdots d_k)) \\
& = \sum_{1 \leq i \leq k} (-1)^{\triangleleft + |c| + \sum_{l=1}^{i-1} |d_l|} \varphi_{k,\beta'}(c_1, \dots, c_k) \otimes g(d_1 \cdots \mu_{1,\beta''}(d_i) \cdots d_k) \\
& + \sum_{0 \leq i \leq k} (-1)^{\triangleleft + \sum_{l=1}^{i-1} (|c_l| + |d_l|)} \varphi_{k+1,\beta'}(c_1, \dots, e_{C_1}, \dots, c_k) \otimes g(d_1 \cdots d_i \cdot \mu_{0,\beta''} \cdots d_k).
\end{aligned}$$

For $k > 0$, the last sum vanishes because φ is unital, therefore the equation is equivalent to

lent to

$$\sum_{\beta' + \beta'' = \beta} (-1)^{|\beta| + |\beta'|} \varphi_{k, \beta'}(c_1, \dots, c_k) \otimes \left[(-1)^k \mu_{1, \beta''}(d_1 \cdots d_k) + \sum_{1 \leq i \leq k} (-1)^{\sum_{l=1}^{i-1} \|d_l\|} d_1 \cdots \mu_{1, \beta_2}(d_i) \cdots d_k \right] = 0.$$

Finally this equation holds because the right hand side of the tensor product vanishes by the last identity in (7). \square

This proposition allows us to give the following

Definition 3.9. Let $\varphi : C_1 \longrightarrow C_2$ and $\psi : D_1 \longrightarrow D_2$ be filtered, unital A_∞ -homomorphisms between filtered dg-algebras C_1, C_2, D_1, D_2 . We define $\varphi \otimes_{dg} \psi : C_1 \otimes_{dg} D_1 \longrightarrow C_2 \otimes_{dg} D_2$ as

$$\varphi \otimes_{dg} \psi = (\varphi \otimes_{dg} id_{D_1}) \circ (id_{C_2} \otimes_{dg} \psi)$$

We would like to point out that this construction is not functorial, that is,

$$(\varphi_1 \otimes_{dg} \psi_1) \circ (\varphi_2 \otimes_{dg} \psi_2) \neq (\varphi_1 \circ \varphi_2) \otimes_{dg} (\psi_1 \circ \psi_2).$$

We believe that these two A_∞ -homomorphism are homotopy equivalent, but we have not tried to prove this. In fact we will not need this statement, instead we will use the following

Lemma 3.10. Let $\varphi : C_1 \longrightarrow C_2$ and $\psi : D_1 \longrightarrow D_2$ be filtered quasi-isomorphisms between filtered dg-algebras C_1, C_2, D_1 and D_2 . Then $\varphi \otimes_{dg} \psi : C_1 \otimes_{dg} D_1 \longrightarrow C_2 \otimes_{dg} D_2$ is a quasi-isomorphism.

PROOF. We only need to check that $(\varphi \otimes_{dg} \psi)_{1,0}$ is an isomorphism on cohomology. For this note that, by definition $(\varphi \otimes_{dg} \psi)_{1,0}(c \otimes d) = \varphi_{1,0}(c) \otimes \psi_{1,0}(d)$. Then the Künneth theorem implies that

$$\varphi_{1,0} \otimes \psi_{1,0} : C_1 \otimes D_1 \longrightarrow C_2 \otimes D_2$$

induces an isomorphism in cohomology. \square

We now tackle the problem for general A_∞ -algebras, A_1, A_2, B_1 and B_2 . In the previous section we constructed a quasi-isomorphism $F_{A_i} : A_i \longrightarrow End_{A_i}$, let $F_{A_i}^{-1}$ be a quasi-inverse. In fact, we could use the homological perturbation lemma to give explicit formulas for $F_{A_i}^{-1}$, but this is unnecessary for our purposes.

Given $\varphi : A_1 \longrightarrow A_2$ and $\psi : B_1 \longrightarrow B_2$ we define $\tilde{\varphi} := F_{A_2} \circ \varphi \circ F_{A_1}^{-1}$ and $\tilde{\psi} := F_{B_2} \circ \psi \circ F_{B_1}^{-1}$. Finally we define $\varphi \otimes_\infty \psi : A_1 \otimes_\infty B_1 \longrightarrow A_2 \otimes_\infty B_2$ as

$$\varphi \otimes_\infty \psi := \tilde{\varphi} \otimes_{dg} \tilde{\psi} : End_{A_1} \otimes_{dg} End_{B_1} \longrightarrow End_{A_2} \otimes_{dg} End_{B_2}.$$

Proposition 3.11. Let $\varphi : A_1 \longrightarrow A_2$ and $\psi : B_1 \longrightarrow B_2$ be quasi-isomorphism. Then $\varphi \otimes_\infty \psi : A_1 \otimes_\infty B_1 \longrightarrow A_2 \otimes_\infty B_2$ is a quasi-isomorphism

PROOF. We only need to check that $(\varphi \otimes_\infty \psi)_{1,0}$ is an isomorphism on cohomology. First note that, if φ and ψ are quasi-isomorphisms then so are $\tilde{\varphi}$ and $\tilde{\psi}$. Thus the statement follows from Lemma 3.10 applied to $\tilde{\varphi}$ and $\tilde{\psi}$. \square

Next we will discuss some monoidal properties of the tensor product of A_∞ -algebras. We won't show that the category of filtered A_∞ -algebras is monoidal, that is probably true but would require a lot more effort to prove and we do not make use of it.

Proposition 3.12. *Let A , B and C be filtered A_∞ -algebras. We have the following:*

1. *Consider the field \mathbb{K} as a filtered A_∞ -algebra with the product as the single non-trivial operation $\mathfrak{m}_{2,0}$. Then $A \otimes_\infty \mathbb{K} \simeq A$.*
2. *$A \otimes_\infty B \simeq B \otimes_\infty A$.*
3. *$A \otimes_\infty (B \otimes_\infty C) \simeq (A \otimes_\infty B) \otimes_\infty C$.*

PROOF.

1. First note that, given a filtered dg-algebra V , we have $V \otimes_{dg} \mathbb{K} \simeq V$. Also, by definition $End_{\mathbb{K}}$ is isomorphic to \mathbb{K} , hence

$$A \otimes_\infty \mathbb{K} = End_A \otimes_{dg} End_{\mathbb{K}} \simeq End_A \otimes_{dg} \mathbb{K} \simeq A,$$

by Lemma 3.10.

2. Given filtered dg-algebras V and W define the naive map

$$\tau : V \otimes_{dg} W \longrightarrow W \otimes_{dg} V$$

by $\tau_{1,0}(v \otimes w) = (-1)^{|v||w|} w \otimes v$. One can easily see this is a A_∞ -isomorphism. Applying this to End_A and End_B we get the desired result.

3. Recall that $End_B \otimes_{dg} End_C \simeq End_{End_B \otimes_{dg} End_C}$, by Corollary 3.4, therefore Lemma 3.10 implies

$$\begin{aligned} A \otimes_\infty (B \otimes_\infty C) &:= End_A \otimes_{dg} End_{End_B \otimes_{dg} End_C} \\ &\simeq End_A \otimes_{dg} (End_B \otimes_{dg} End_C). \end{aligned}$$

Similarly

$$(A \otimes_\infty B) \otimes_\infty C \simeq (End_A \otimes_{dg} End_B) \otimes_{dg} End_C.$$

Now, the result follows from the general statement: given filtered dg-algebras V , W and U , the naive map

$$id_{1,0} : V \otimes_{dg} (W \otimes_{dg} U) \longrightarrow (V \otimes_{dg} W) \otimes_{dg} U$$

is an A_∞ -isomorphism.

□

4 Another model for $A \otimes_\infty B$

In this section, given two A_∞ -algebras A and B , we will use the homological perturbation lemma to construct an A_∞ -algebra structure on the vector space $A \otimes B$ quasi-isomorphic to $A \otimes_\infty B$.

We start by introducing some terminology and constructions relating to trees. Denote by G_n the set of trees with n leaves and $val(v) \geq 2$ for each internal vertex v . Given a vertex v we will denote by $v(i)$ the i -th incident edge at v .

For each pair $x, y \in V(T) \cup E(T)$ we say x is under y if the unique path P_x in T from x to the root of T contains y . Note that this defines a partial ordering. Suppose x and y are not comparable and denote by w the first (furthest from the root) vertex that P_x and P_y have in common. Then P_x and P_y contain edges $w(i)$ and $w(j)$ respectively. We say that x is to the left of y if $i < j$.

Given $x \in V(T) \cup E(T)$ we denote by p_x (respectively q_x) the left (respectively right) most leaf of T under x . Additionally, given $e \in E(T)$ we define integers J_e , I_e and \tilde{I}_e to be zero when e is a leaf and when e is not a leaf, J_e is the number of leaves under e , \tilde{I}_e is the number of internal edges under e and $I_e = \tilde{I}_e + 1$.

Next, we denote by $G_n^{bin} \subset G_n$ the subset of binary trees, that is $val(v) = 2$ for all internal vertices. There is a partial order on the set G_n^{bin} , which was introduced in [10]. It is generated by the relation on the following figure:

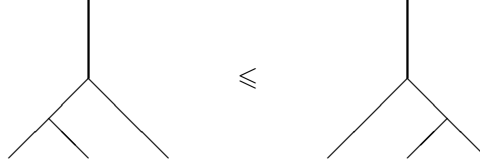


Figure 2: Relation that generates the partial order on binary trees

This means we say $T_1 \leq T_2$ if there is a sequence of trees starting at T_1 and ending in T_2 , such that consecutive trees are equal to each other except in the neighborhood of a vertex where they look like Figure 2. There are absolute minimum and maximum for this order. The minimum is the binary tree with all internal edges right leaning, which we denote by \underline{M}_n . The maximum has all internal edges left leaning and it is denoted by \overline{M}_n .

Given $T \in G_n$ we define $T_{max} \in G_n^{bin}$ as the maximal (with respect to the order just described) binary tree that resolves T . That is, T can be obtained by collapsing several edges in T_{max} . See Figure 3 for an example.

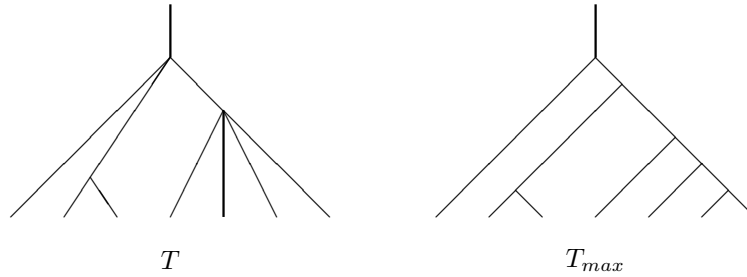


Figure 3: Example of maximal tree resolving T

Similarly, we define T_{min} as the minimal binary tree that resolves T . We denote by $c_n \in G_n$ the tree with only one internal vertex, and call it the n -corolla.

We introduce one more subset of G_n : we denote by $L_n \subset G_n$ the subset of trees obtained by grafting corollas together but avoiding the last leaf. More formally, $T \in L_n$ if $T = c_n$ or there exist integers $1 \leq i_1 < \dots < i_l < m$ and trees $T_j \in L_{n_j}$ such that

$$T = c_m \circ_{i_1, \dots, i_l} (T_1, \dots, T_l),$$

where the notation \circ_{i_1, \dots, i_l} stands for grafting the root of T_j to the i_j -th leaf of c_m .

Now let $V \in L_n$ with k internal edges. There is an obvious correspondence between internal edges of V and right leaning (internal) edges of V_{max} . Denote by $R(B)$ the set of right leaning internal edges of a binary tree B and by $|R(B)|$ the order of this set. We

define

$$\alpha(V) := \sum_{\substack{S \in G_n^{bin} \\ S \geq V_{max} \\ |R(S)|=k}} S/R(S),$$

where $S/R(S)$ is the tree obtained by collapsing all the edges of S in $R(S)$. For example Note that $\alpha(V)$ is non-zero as it always includes $\alpha_0(T) := T_{max}/R(T_{max})$.

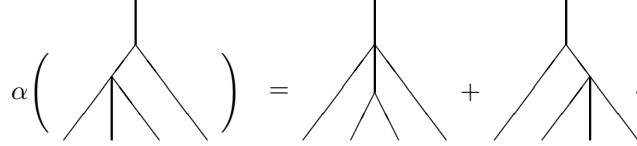


Figure 4: Example of α

Finally, given $U \in G_n$ and $\beta \in G$ denote by U_β the sum of all elements of $\Gamma_n(\beta)$ whose underlying tree is U . We use U_β as a flow chart to define a map

$$U_\beta^A : A^{\otimes k} \longrightarrow A,$$

by assigning to each $v \in V(U_\beta)$ the map $\mathbf{m}_{val(v), \nu(v)}^A$. Similarly we define maps U_β^B .

Theorem 4.1. *Let A and B be unital filtered A_∞ -algebras. Then the tensor product $A \otimes_\infty B$ is quasi-isomorphic to the A_∞ -algebra $(A \otimes B, \mathbf{m}^\otimes)$ with operations given by $\mathbf{m}_{k, \beta}^\otimes = \sum_{\beta_1 + \beta_2 = \beta} \mathbf{m}_{k, \beta_1 \times \beta_2}^\otimes$, where*

$$\mathbf{m}_{0, \beta_1 \times \beta_2}^\otimes = \begin{cases} \mathbf{m}_{0, \beta_1}^A \otimes e_B & , \beta_2 = 0 \\ e_A \otimes \mathbf{m}_{0, \beta_2}^B & , \beta_1 = 0 \\ 0 & , \beta_1, \beta_2 \neq 0 \end{cases},$$

$$\mathbf{m}_{1, \beta_1 \times \beta_2}^\otimes = \begin{cases} \mathbf{m}_{1, \beta_1} \otimes id & , \beta_2 = 0 \\ id \otimes \mathbf{m}_{1, \beta_2} & , \beta_1 = 0 \\ 0 & , \beta_1, \beta_2 \neq 0 \end{cases}, \quad (8)$$

$$\mathbf{m}_{n, \beta_1 \times \beta_2}^\otimes(a_1 \otimes b_1, \dots, a_n \otimes b_n) = \sum_{T \in L_n} (-1)^\epsilon T_{\beta_1}^A(a_1, \dots, a_n) \otimes \alpha(T)_{\beta_2}^B(b_1, \dots, b_n).$$

The sign is defined as $\epsilon = \theta(T) + \triangleleft + |\mathbf{a}| |E_{int}(T)| + \gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}$, with

$$\gamma_{\mathbf{a}} = \sum_{v \in V(T)} \sum_{i < p_v} |a_i|, \quad \gamma_{\mathbf{b}} = \sum_{v \in V(\alpha(T))} \sum_{i < p_v} |b_i|,$$

$$\theta(T) = \sum_{v \in V(T)} \left(\sum_i ((val(v) - i - 1) I_{v(i)} + \tilde{I}_{v(i)} + J_{v(i)}) + \sum_{i < j} I_{v(i)} (\tilde{I}_{v(j)} + J_{v(j)}) \right).$$

To illustrate the theorem, in Figure 5 we describe explicitly \mathbf{m}_3^\otimes and \mathbf{m}_4^\otimes , indicating the sign $\theta(T)$.

Assume A and B are classical A_∞ -algebras, then (8) agrees exactly with the formula for $\mathbf{m}_n^{\otimes su}$ provided in [10, Theorem 3], up to sign. The difference in the sign comes from the fact that Markl and Shnider use a different sign rule in the definition of A_∞ -algebra. In (1), they take $*$ = $i(j+1) + jn + j(|a_1| + \dots + |a_{i-1}|)$. This definition is equivalent to ours via the change

$$\mathbf{m}'_k(a_1, \dots, a_n) \mapsto (-1)^{\sum (k-i)|a_i|} \mathbf{m}_k(a_1, \dots, a_n).$$

Taking this sign change in consideration we can show that (8) agrees with the formula in [10]. Therefore we obtain the following

$$\begin{aligned}
\mathfrak{m}_3^\otimes &= \text{Diagram 1} \otimes \text{Diagram 2} + \text{Diagram 3} \otimes \text{Diagram 4}, \\
\mathfrak{m}_4^\otimes &= \text{Diagram 5} \otimes \text{Diagram 6} - \text{Diagram 7} \otimes \text{Diagram 8} - \text{Diagram 9} \otimes \text{Diagram 10} + \\
&+ \text{Diagram 11} \otimes \text{Diagram 12} - \text{Diagram 13} \otimes \text{Diagram 14} + \text{Diagram 15} \otimes \text{Diagram 16}.
\end{aligned}$$

Figure 5: Formulas for \mathfrak{m}_3^\otimes and \mathfrak{m}_4^\otimes .

Corollary 4.2. *Let A and B be classical A_∞ -algebras, then the tensor product $A \otimes_\infty B$ is quasi-isomorphic to the tensor product $A \otimes_{SU} B$ defined by Markl and Shnider in [10].*

Before proving the theorem, we will give two alternative descriptions of $\mathfrak{m}_{n,\beta_1 \times \beta_2}^\otimes$ for $n \geq 3$.

In the same way we defined $L_n \subseteq G_n$ we can define $R_n \subseteq G_n$ as the set of trees obtained by grafting corollas together but avoiding the first leaf. Also, given $T \in R_n$ with k internal edges we define

$$\bar{\alpha}(T) = \sum_{\substack{W \in G_n^{bin} \\ W \leq T_{min} \\ |L(W)|=k}} W/L(W),$$

where $W/L(W)$ is obtained by collapsing all the edges in $L(W)$, the set of left leaning edges on W . We have the following:

Proposition 4.3. *Let $n \geq 3$, then we have the alternative descriptions*

$$\mathfrak{m}_{n,\beta_1 \times \beta_2}^\otimes = \sum_{T \in R_n} \pm \bar{\alpha}(T)_{\beta_1}^A \otimes T_{\beta_2}^B,$$

and

$$\mathfrak{m}_{n,\beta_1 \times \beta_2}^\otimes = \sum_{\substack{|E_{int}(U)| + |E_{int}(W)| = n-2 \\ U_{max} \leq W_{min}}} \pm U_{\beta_1}^A \otimes W_{\beta_2}^B.$$

PROOF. Easy combinatorial check, see [10]. □

Proof of Theorem 4.1: This is a direct application of the homological perturbation lemma. Consider the chain complex $(A \otimes B, \delta = \mathfrak{m}_{1,0}^A \otimes id + id \otimes \mathfrak{m}_{1,0}^B)$ and define maps

$$\begin{aligned}
i &= F_{1,0}^A \otimes F_{1,0}^A : A \otimes B \longrightarrow End_A \otimes End_B, \\
P &= P_A \otimes P_B : End_A \otimes End_B \longrightarrow A \otimes B,
\end{aligned} \tag{9}$$

$$H = \Pi_A \otimes H_B + H_A \otimes id : End_A \otimes End_B \longrightarrow End_A \otimes End_B, \tag{10}$$

where $\Pi_A = F_{1,0}^A \circ P_A$ and P_A, H_A are as defined in Lemma 3.3.

Lemma 4.4. *The maps i, P and H determine homotopy data that satisfies the side conditions.*

PROOF. The maps i and P are chain maps because they are the tensor product of chain maps. Next, by direct computation we have

$$\begin{aligned}\mu_{1,0}^\otimes H + H\mu_{1,0}^\otimes &= \Pi_A \otimes \mu_{1,0}^B H_B - H_A \otimes \mu_{1,0}^B + \mu_{1,0}^A \Pi_A \otimes H_B + \mu_{1,0}^A H_A \otimes id + \\ &\quad + \Pi_A \otimes H_B \mu_{1,0}^B + H_A \otimes \mu_{1,0}^B - \Pi_A \mu_{1,0}^A \otimes H_B + H_A \mu_{1,0}^A \otimes id \\ &= \Pi_A \otimes (\Pi_B - id) + (\mu_{1,0}^A \Pi_A - \Pi_A \mu_{1,0}^A) \otimes H_B + (\Pi_A - id) \otimes id \\ &= \Pi_A \otimes \Pi_B - id = i \circ P - id,\end{aligned}$$

using Lemma 3.3 for A and B and the fact that Π_A is a chain map.

The side conditions follow from direct computations that we omit. Finally we observe that $i(e_A \otimes e_B) = id$. \square

Now Theorem 2.10, provides an A_∞ -algebra structure on $A \otimes B$ with $\eta_{1,0} = \delta$. By Proposition 2.11 this A_∞ -algebra is unital and there is a quasi-isomorphism $\varphi : (A \otimes B, \nu) \longrightarrow A \otimes_\infty B$. Moreover we have explicit formulas

$$\eta_{n,\beta} = \sum_{T \in \Gamma_n(\beta)} \eta_T.$$

In this situation, since the A_∞ -algebras are bi-gapped we can further decompose the expression for $\eta_{n,\beta}$ as

$$\eta_{n,\beta} = \sum_{\beta_1 + \beta_2 = \beta} \eta_{n,\beta \times \beta_2} = \sum_{T \in \Gamma_n(\beta_1 \times \beta_2)} \eta_T \quad (11)$$

All that is left to prove now is that (11) gives the formulas in (8). This will be done in several lemmas in which we will determine the trees $T \in \Gamma_n(\beta_1 \times \beta_2)$ that give nontrivial contributions and will describe those. The first lemma follows from easy computations that we omit.

Lemma 4.5. *Let (C, \mathfrak{m}) be a G -gapped filtered A_∞ -algebra and consider $\beta \in G$ and $f, g \in \text{End}_C$. We have the following identities*

1. $H_C(\mu_{0,\beta}) = 0$
2. $H_C \circ \mu_{2,0}(f, H_C(g)) = 0$, $P_C \circ \mu_{2,0}(f, H_C(g)) = 0$.
3. $H_C \circ \mu_{1,\beta} \circ H_C = 0$, $P_C \circ \mu_{1,\beta} \circ H_C = 0$, if $\beta \neq 0$.
4. $[H_C \circ \mu_{1,\beta} \circ F_{1,0}^C(\xi)]_*(u, c_1, \dots, c_r) = \mathfrak{m}_{r+2,\beta}(\xi, u, c_1, \dots, c_r)$.
5. $P_C \circ \mu_{1,\beta} \circ F_{1,0}^C(\xi) = \mathfrak{m}_{1,\beta}(\xi)$.

Lemma 4.6. $H(\mu_{0,\beta_1 \times 0}^\otimes) = H(\mu_{0,0 \times \beta_2}^\otimes) = 0$. Therefore $\eta_T = 0$, whenever T has a vertex of valency zero.

PROOF. From the definitions we have,

$$H(\mu_{0,\beta_1 \times 0}^\otimes) = \Pi_A(\mu_{0,\beta_1}) \otimes H_B(id_{\text{End}_B}) + H_A(\mu_{0,\beta_1}) \otimes id_{\text{End}_B} = 0 \quad (12)$$

by part (1) of Lemma 4.5 and the easy fact $H_B(id_{\text{End}_B}) = 0$. The same argument shows $H(\mu_{0,0 \times \beta_2}^\otimes) = 0$.

For the second statement, recall that, when defining η_T a vertex of valency zero would be assigned with $\mu_{0,\beta}$ and the outgoing edge with H . But we just showed that composition is zero, thus $\eta_T = 0$. \square

This lemma is enough to describe $\eta_{0,\beta}$. Given $T \in \Gamma_0(\beta_1 \times \beta_2)$ such that $\eta_T \neq 0$, it cannot have internal edges by Lemma 4.6. Therefore T has to be a tree with one single internal vertex of weight $\beta_1 \times 0$ or $0 \times \beta_2$. In these cases we have

$$\eta_{0,\beta_1 \times 0} = P(\mu_{0,\beta_1 \times 0}) = P_A(\mu_{0,\beta_1}) \otimes P_B(Id_B) = \mathfrak{m}_{0,\beta_1} \otimes e_B.$$

and similarly $\eta_{0,0 \times \beta_2} = e_A \otimes \mathfrak{m}_{0,\beta_2}$.

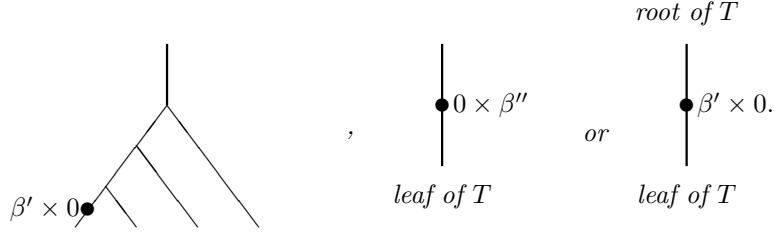
Let us now consider the general case, $\eta_{n,\beta} = \sum_{T \in \Gamma_n(\beta)} \eta_T$. First note that, since $A \otimes_\infty B$ is a filtered dg-algebra, the vertices of T have valency ≤ 2 . By Lemma 4.6 it cannot have vertices of valency zero. If a vertex has valency one, we say the incoming edge is vertical, if it has valency two, we say the first incoming edge is right leaning and the second is left leaning.

In order to better describe the maps η_T , we will further decompose η_T into the following sum

$$\eta_{n,\beta} = \sum_{T \in \Gamma_n(\beta)} \eta_T = \sum_{\substack{T \in \Gamma_n(\beta) \\ T_H}} \eta_{T_H}.$$

Here η_{T_H} is defined in the same way as η_T except we assign either $\Pi_A \otimes H_B$ or $H_A \otimes id$ to each internal edge of T and the sum is taken over the $2^{|E_{int}(T)|}$ ways of doing this.

Lemma 4.7. *Given $T \in \Gamma_n(\beta_1 \times \beta_2)$ and T_H as above, cut T along the edges with $\Pi_A \otimes H_B$ assigned. Denote by W_1, \dots, W_{k+1} the trees obtained. If $\eta_{T_H} \neq 0$ then each of the W_i is equal to one of the following:*



In the first case there are $l \geq 2$ leaves and if $\beta' = 0$, we remove the vertex of valency one. In the second and third cases, we must have $\beta', \beta'' \neq 0$.

PROOF. Let W be one such tree, the root of W either coincides with the root of T , in which case it is assigned with P , or it is assigned with $\Pi_A \otimes H_B$. Denote by η_W the map induced by W , we are interested in computing $\eta_W(\xi_1, \dots, \xi_k)$ for $\xi_i = f_i \otimes g_i$ where each $f_i \otimes g_i = \Pi_A(\tilde{f}_i) \otimes H_B(\tilde{g}_i)$ or $f_i \otimes g_i = i(a_i \otimes b_i)$, since each leaf of W is either a leaf of T_H or an internal edge of T_H labeled by $\Pi_A \otimes H_B$. By construction, all the internal edges of W are labeled with $H_A \otimes id$.

Claim: All the internal edges of W are right leaning.

- Suppose there was a left leaning internal edge. If it wasn't adjacent to the top vertex of W , it would lead to a composition of the form

$$(H_A \otimes id) \circ \mu_{2,0}^\otimes(\rho \otimes \tau, H_A(\varphi) \otimes \psi) = H_A(\mu_{2,0}(\rho, H_A(\varphi))) \otimes \mu_{2,0}(\tau, \psi) = 0,$$

which vanishes by part 2 of Lemma 4.5. If the edge is adjacent to the top vertex of W , we have one of the compositions: $(\Pi_A \otimes H_B) \circ \mu_{2,0}^\otimes(\rho \otimes \tau, H_A(\varphi) \otimes \psi)$ or $(P_A \otimes P_B) \circ \mu_{2,0}^\otimes(\rho \otimes \tau, H_A(\varphi) \otimes \psi)$, depending whether the root of W coincides with the root of T or not. In both cases we get zero by part 2 of Lemma 4.5.

- If there was a vertical interior edge, the endpoint of that edge would have weight $\beta' \times 0$ or $0 \times \beta''$. In the first case, we would have one of the compositions $H_A \circ$

$\mu_{1,\beta'} \circ H_A$ or $P_A \circ \mu_{1,\beta'} \circ H_A$, which are both zero by part 3 of Lemma 4.5. The second case would result in compositions of the form $H_B \circ H_B$ or $P_B \circ H_B$, which are both zero by the side conditions.

This claim immediately implies that W can have at most one vertex of valency one and it must be adjacent to the first leaf of W . Moreover, when W has $k > 1$ leaves, T is the minimal binary tree, once we forget the vertex of valency one. When $k = 1$ then W has a single internal vertex.

Next we will see that, if the vertex of valency one has weight $\beta' \times 0$ and $k = 1$, then $W = T$. If the leaf of W was not a leaf of T , the input would be of the form $\Pi_A(\tilde{f}_i) \otimes H_B(\tilde{g}_i)$, which leads to one of the compositions $H_B \circ H_B$ or $P_B \circ H_B$, that vanish as a consequence of the side conditions. Similarly, if the root of W did not coincide with the root of T we would get $(\Pi_A \otimes H_B)(\mu_{1,\beta'} \otimes id) \circ i$, which again vanishes by the side conditions.

Now consider the case when the vertex of valency one has weight $0 \times \beta''$. If the leaf adjacent to this vertex was not a leaf of T we would obtain either $(\Pi_A \otimes H_B)(id \otimes \mu_{1,\beta''})(\Pi_A \otimes H_B)$ or $(P_A \otimes P_B)(id \otimes \mu_{1,\beta''})(\Pi_A \otimes H_B)$. Again, the side conditions imply that both vanish. \square

We are now ready to describe $\eta_{1,\beta_1 \times \beta_2}$. Consider $T \in \Gamma_1(\beta_1 \times \beta_2)$ and let T_H be as above. Then T_H must be obtained by gluing trees of the second and third types in the lemma. But then it must equal one of these, since the leaves of these trees must also be leaves of T . Therefore

$$\begin{aligned} \eta_{1,\beta_1 \times 0}(a \otimes b) &= (p_A \otimes p_B) \circ (\mu_{1,\beta_1} \otimes id) \circ (i(a) \otimes i(b)) \\ &= \mathbf{m}_{1,\beta_1}(a) \otimes b \end{aligned}$$

by part 5 of Lemma 4.5. Similarly $\eta_{1,0 \times \beta_2}(a \otimes b) = (-1)^{|a|} a \otimes \mathbf{m}_{1,\beta_2}(b)$ and $\eta_{1,\beta_1 \times \beta_2} = 0$ when $\beta_1, \beta_2 \neq 0$.

To describe $\eta_{n,\beta} = \sum_{T \in \Gamma_n(\beta)} \eta_T$, for $n \geq 2$ we need two more lemmas.

Lemma 4.8. *Let $T \in \Gamma_n(\beta_1 \times \beta_2)$ for $n \geq 2$ and consider the decomposition of $T_H = W_1 \cup \dots \cup W_{k+1}$ provided by Lemma 4.7. If $\eta_{T_H} \neq 0$, then each of the W_i is never grafted to the last leaf of another W_j .*

PROOF. If some W_i was grafted to the last leaf of another W_j it would lead to either a composition of the form

$$(\Pi_A \otimes H_B) \circ \mu_{2,0}^{\otimes}(\rho \otimes \tau, \Pi_A(\varphi) \otimes H_B(\psi))$$

or $(P_A \otimes P_B) \circ \mu_{2,0}^{\otimes}(\rho \otimes \tau, \Pi_A(\varphi) \otimes H_B(\psi))$, depending whether the root of W_j coincides with the root of T or not. In both cases we get zero by part 2 of Lemma 4.5. \square

Lemma 4.9. *Let W be a tree of the first type in Lemma 4.7 with k leaves. For each $1 \leq i \leq k$, consider $\xi_i = i_A(a_i) \otimes f_i$ for $a_i \in A$ and $f_i \in \text{End}_B$. If the root of W does not coincide with the root T , then*

$$\eta_W(\xi_1, \dots, \xi_k) = (-1)^{\alpha + \triangleleft} i_A(\mathbf{m}_{k,\beta}(a_1, \dots, a_k)) \otimes H'_B(f_1 \circ \dots \circ f_k), \quad (13)$$

where $H'(\tau) = (-1)^{|\tau|} H(\tau)$ and the sign $\alpha = k + \sum_{i=1}^k |a_i| + (k - i - 1)|f_i|$. If the root of W does coincide with the root of T , then

$$\eta_W(\xi_1, \dots, \xi_k) = (-1)^{\alpha' + \triangleleft} \mathbf{m}_{k,\beta}(a_1, \dots, a_k) \otimes f_1 \circ \dots \circ f_k(e_B), \quad (14)$$

with $\alpha' = \sum_{i=1}^k (k-i-1)|f_i|$.

When W is a tree of second type (in Lemma 4.7), we have

$$\eta_W(a \otimes b) = (-1)^{|a|} F_{1,0}^A(a) \otimes F_{1,\beta''}^B(b).$$

PROOF. This is a straightforward computation that we leave to the reader. \square

Given $T_H \in \Gamma_n(\beta_1 \times \beta_2)$, $n \geq 2$, consider the decomposition

$$T_H = S_1 \cup \dots \cup S_{t+1} \cup R_1 \cup \dots \cup R_j \quad (15)$$

provided by Lemma 4.7, where each S_i is of the first type and R_l is of the second type (in the same lemma). Recall that each S_i has at most a vertex of valency one with weight $\beta'_i \times 0$ and each R_l has a unique vertex of weight $0 \times \beta''_l$. Moreover, $\sum_{i=1}^k \beta'_i = \beta_1$ and $\sum_{i=1}^k \beta''_i = \beta_2$.

Let U be the tree obtained from T_H by forgetting all the subtrees R_l and replacing each S_i by the corolla of the respective size with β'_i assigned to its unique vertex. Then U has t internal edges and by Lemma 4.8, $U \in L_n$. Observe that U_{max} can be obtained from T_H by forgetting the R_l and replacing each S_i by the maximal binary tree with the same number of leaves.

Applying Lemma 4.9 to all subtrees S_i and R_j we conclude

$$\eta_{T_H}(a_1 \otimes b_1, \dots, a_n \otimes b_n) = (-1)^\dagger U^A(a_1, \dots, a_n) \otimes \psi(b_1, \dots, b_n) \quad (16)$$

for some sign \dagger which we will describe later and for a map ψ which we now describe.

We consider first, the case when there is a single S . In this case, by Lemma 4.8, there can be at most $n-1$ trees of second type R_1, \dots, R_{n-1} grafted to the first $n-1$ leaves of S . An easy computation using Part 4) of Lemma 4.5 shows that

$$\psi(b_1, \dots, b_n) = \mathbf{m}_{2,\beta''_1}(b_1, \mathbf{m}_{2,\beta''_2}(b_2, \dots, \mathbf{m}_{2,\beta''_{n-1}}(b_{n-1}, b_n) \dots)). \quad (17)$$

If there are several S_i 's we order them so that whenever S_i is above S_j , $i < j$. Then the root of S_1 agrees with the root of T . If we denote by l the number of leaves of S_1 , by definition there are $\varphi_m, \tau_k \in \text{End}_B$ so that

$$\psi(b_1, \dots, b_n) = (\varphi_1 \circ \dots \circ \varphi_l)_0(e_B).$$

Lemma 4.9 implies that for each i , $\varphi_i = H'(\tau_1 \circ \dots \circ \tau_{i_0})$ or $\varphi_i = F_{1,\beta_{i_0}}^B(b_{i_0})$, moreover, we must have $\varphi_l = F_{1,0}^B(b_n)$, by Lemma 4.8.

Let j be the minimum such that $\varphi_i = F_{1,\beta_{i_0}}^B(b_{i_0})$ for $i > j$. Then, proceeding as in the calculation of (17) we have

$$\begin{aligned} \psi(b_1, \dots, b_n) &= [\varphi_1 \circ \dots \circ \varphi_{j-1} \circ H'(\tau_1, \dots, \tau_{j_0})](\mathbf{m}_{2,\beta_q}(b_q, \mathbf{m}_{2,\beta_{q+1}}(b_{q+1}, \dots, \mathbf{m}_{2,\beta_{n-1}}(b_{n-1}, b_n) \dots)) \dots) \\ &= \varphi_1 \circ \dots \circ \varphi_{j-1} \left(\tau_1 \circ \dots \circ \tau_j (b_{q-1}, \mathbf{m}_{2,\beta_q}(b_q, \dots, \mathbf{m}_{2,\beta_{n-1}}(b_{n-1}, b_n) \dots)) \right), \end{aligned}$$

since, again by Lemma 4.8, we must have $\tau_{j_0} = F_{1,0}^B(b_{q-1})$.

Repeating this argument until we exhaust all of the S_i and using the formula

$$(\varphi_1 \circ \dots \circ \varphi_l)_r(v; y_1, \dots, y_r) = \sum_{0 \leq i_l \leq i_{l-1} \leq \dots \leq i_1 = r} \varphi_1(\dots \varphi_{l_1}(\varphi_l(v; y_1, \dots, y_{i_l}); \dots y_{i_{l-1}}) \dots), \quad (18)$$

we conclude

$$\psi(b_1, \dots, b_n) = \sum F_{1,\beta''_1}^B(b_1)(\dots F_{1,\beta''_l}^B(b_l)(b_{l+1} \dots b_q) \dots). \quad (19)$$

The terms in the sum are in one to one correspondence with binary trees Z with t right leaning internal edges and $U_{max} \leq Z$. This can be seen as follows: each of the t right leaning internal edges in U_{max} corresponds to one of the internal edges of T_H labeled with $\Pi_A \otimes H_B$ and therefore to applying H' when computing ψ . Each time we apply H' , we have to sum over all the ways of composing elements in End_B using (18). These are in one to one correspondence with all the trees W that can be obtained from U_{max} by performing a sequence of moves described in Figure 2, as long as the edge being moved is an internal edge, which are exactly the ones with t right leaning internal edges and $U_{max} \leq Z$. For example, U_{max} corresponds to taking $0 = i_l = \dots = i_2$, $i_1 = r$ in (18), every time.

Given such tree Z , we collapse all the right leaning edges and obtain a tree Q . For each vertex v in Q , let p_v be the left most leaf under v , we define the weight of v to be β_l'' if p_v is contained in R_l (for some l) and zero otherwise. Then the corresponding term in the sum (19) is exactly $Q^B(b_1, \dots, b_n)$.

Thus we conclude that

$$\eta_{T_H}(a_1 \otimes b_1, \dots, a_n \otimes b_n) = (-1)^{\dagger} U^A(a_1, \dots, a_n) \otimes \alpha(U)^B(b_1, \dots, b_n). \quad (20)$$

The only thing left to show is that \dagger coincides with the sign described in (8). From now on all the computations are modulo 2.

Each of the trees S_i contributes to \dagger with the sign described in Lemma 4.9. By Lemma 4.8, the last leaf of each S_i coincides with one leaf of T , which we denote by m_i . This contributes with the sign $(-1)^{|b_{m_i}|}$, coming from the computation $F_{1,0}^B(b_{m_i})(e_B) = (-1)^{|b_{m_i}|} b_{m_i}$. Therefore, noting that each S_i corresponds to one vertex in U , we have

$$\begin{aligned} \dagger = & \sum_{v \in V(U)} \left[val(v) + \sum_{i < j} |\tilde{f}_i| |\tilde{a}_j| + \sum_{i=1}^{val(v)} |\tilde{a}_i| + (val(v) + 1 - i) |\tilde{f}_i| \right] \\ & + \left(\sum_{i=1}^n |a_i| + n + |E_{int}(U)| \right) + \sum_{i=1}^{t+1} |b_{m_i}|, \end{aligned} \quad (21)$$

where

$$|\tilde{f}_i| = \sum_{p_{v(i)} \leq s \leq q_{v(i)}} |b_s| + I_{v(i)} \quad \text{and} \quad |\tilde{a}_i| = \sum_{p_{v(i)} \leq s \leq q_{v(i)}} |a_s| + \tilde{I}_{v(i)} + J_{v(i)}.$$

We will divide $\dagger = \mathcal{A} + \mathcal{B} + \mathcal{R}$ into three sums, the first involving terms with $|a_m|$, the second involving terms with $|b_m|$ and the third one with the remaining terms. Explicitly we have

$$\begin{aligned} \mathcal{A} = & \sum_{v \in V(U)} \left[\sum_{i < j} I_{v(i)} \left(\sum_{p_{v(j)} \leq s \leq q_{v(j)}} |a_s| \right) + \sum_{p_v \leq s \leq q_v} |a_s| \right] + \sum_{i=1}^n |a_i| \\ \mathcal{B} = & \sum_{v \in V(U)} \left[\sum_{i < j} |b_i| (\tilde{I}_{v(j)} + J_{v(j)}) + \sum_i (val(v) - 1 - i) \sum_{p_{v(i)} \leq s \leq q_{v(i)}} |b_s| \right] + \sum_{i=1}^k |b_{m_i}| \\ \mathcal{R} = & \sum_{v \in V(U)} \left(\sum_i ((val(v) - i - 1) I_{v(i)} + \tilde{I}_{v(i)} + J_{v(i)}) + \sum_{i < j} I_{v(i)} (\tilde{I}_{v(j)} + J_{v(j)}) \right) \\ & + \sum_{v \in V(U)} val(v) + n + |E_{int}(U)| \\ & + \sum_{v \in V(U)} \sum_{i < j} \left(\sum_{p_{v(i)} \leq s \leq q_{v(i)}} |b_s| \right) \left(\sum_{p_{v(i)} \leq r \leq q_{v(i)}} |a_r| \right). \end{aligned}$$

First observe that $\mathcal{R} = \theta(U) + \triangleleft$, because $\sum_{v \in V(U)} \text{val}(v) + n + |E_{\text{int}}(U)| = 0$. Next we claim that $\mathcal{A} = |\mathbf{a}| |E_{\text{int}}(U)| + \gamma_{\mathbf{a}}$. This claim is equivalent to

$$\sum_{v \in V(U)} \left[\sum_{i < j} I_{v(i)} \left(\sum_{p_{v(j)} \leq s \leq q_{v(j)}} |a_s| \right) + \sum_{s \leq q_v} |a_s| \right] + (|E_{\text{int}}(U)| + 1) |\mathbf{a}| = 0.$$

Noting that U has $|E_{\text{int}}(U)| + 1$ internal vertices, this is in turn equivalent to

$$\sum_{v \in V(U)} \sum_{s > q_v} |a_s| = \sum_{v \in V(T)} \sum_{i < j} I_{v(i)} \left(\sum_{p_{v(j)} \leq s \leq q_{v(j)}} |a_s| \right).$$

Now this statement follows from the fact that both sides equal

$$\sum_{i=1}^n |a_i| \nu_i,$$

where ν_i is the number of internal vertices (or internal edges) of U to the left of the i -th leaf.

At last we show that $\mathcal{B} = \gamma_{\mathbf{b}}$. As above we can give an alternative description of $\gamma_{\mathbf{b}}$, namely

$$\gamma_{\mathbf{b}} = \sum_{i=1}^n |b_i| \bar{\nu}_i,$$

where $\bar{\nu}_i$ is the number of internal vertices in $\alpha_0(U)$ to the right of the i -th leaf. If we exclude the internal vertex adjacent to the root (which is not to the right of any leaf), internal vertices in $\alpha_0(U)$ are in one-to-one correspondence with the left leaning edges in U_{max} . These correspond to the internal edges of the S_m in (15). On the other hand we can rewrite \mathcal{B} as

$$\mathcal{B} = \sum_{v \in V(T)} \sum_{i < j} |b_i| (\tilde{I}_{v(j)} + J_{v(j)}) + \sum_i^{\text{val}(v)-1} (\text{val}(v) - 1 - i) \sum_{p_{v(i)} \leq s \leq q_{v(i)}} |b_s|.$$

If we denote by S_v the subtree in T_H corresponding to $v \in V(U)$, then $(\text{val}(v) - 1 - i)$ is the number of internal edges of S_v to the right of its i -th leaf. Also $(\tilde{I}_{v(j)} + J_{v(j)})$ is the number of internal edges of the other $S_{v'}$, under $v(j)$. Therefore we conclude that

$$\mathcal{B} = \sum_{i=1}^n |b_i| \bar{\nu}_i = \gamma_{\mathbf{b}}.$$

This finishes the proof of Theorem 4.1. □

5 Criterion for tensor product

In this section we will describe a set of conditions that imply that a given filtered A_{∞} -algebra is the tensor product of two subalgebras. Although rather restrictive this criterion will be sufficient for the two applications we have in mind: the description of deformations of a tensor product via bounding cochains in Section 6 and the proof of the Künneth Theorem for the Fukaya algebra of the product of two Lagrangian submanifolds in [2].

We start by defining subalgebra and commuting subalgebras.

Definition 5.1. Let (A, \mathfrak{m}^A) and (C, μ) be (respectively) G_A and G -gapped filtered A_∞ -algebras, for discrete submonoids $G_A \subseteq G$. We say A is a *subalgebra* of C if $A \subseteq C$, $e_A = e_C$ and for all $k > 0$ and $a_1, \dots, a_k \in A$ we have

$$\begin{aligned}\mu_{k,\beta}(a_1, \dots, a_k) &= \mathfrak{m}_{k,\beta}^A(a_1, \dots, a_k), \quad \beta \in G_A, \\ \mu_{k,\beta}(a_1, \dots, a_k) &= 0, \quad \beta \in G \setminus G_A.\end{aligned}$$

Definition 5.2. Let (A, \mathfrak{m}^A) and (B, \mathfrak{m}^B) be G_A and G_B -gapped filtered A_∞ -algebras. Suppose A and B are subalgebras of (C, μ) a G -gapped A_∞ -algebra with $G = G_A + G_B$. Denote by $K : A \otimes B \rightarrow C$ the map defined as $K(a \otimes b) = (-1)^{|a|} \mu_{2,0}(a, b)$. We say A and B are *commuting subalgebras* if given $c = K(a \otimes b)$ and $c_1, \dots, c_k \in C$ such that for each i , $c_i = a_i$ or $c_i = b_i$ for some $a_i \in A$ and $b_i \in B$, the following conditions hold.

1. For $k > 0$, $\mu_{k,\beta}(c_1, \dots, c_k) = 0$ unless
 - (a) $(k, \beta) = (2, 0)$ and $c_1 \in A$, $c_2 \in B$ (or vice-versa) in which case, $\mu_{2,0}(c_1, c_2) + (-1)^{\|c_1\| \|c_2\|} \mu_{2,0}(c_2, c_1) = 0$,
 - (b) $c_i = a_i$ for all i and $\beta \in G_1$,
 - (c) $c_i = b_i$ for all i and $\beta \in G_2$.
2. $\mu_{0,\beta} = \mathfrak{m}_{0,\beta}^A + \mathfrak{m}_{0,\beta}^B$, with the convention that $\mathfrak{m}_{0,\beta}^A = 0$ (respectively $\mathfrak{m}_{0,\beta}^B$) if $\beta \notin G_A$ (respectively $\beta \notin G_B$).
3. $\mu_{k+1,\beta}(c_1, \dots, c_i, c, c_{i+1}, \dots, c_k) = 0$ unless
 - (a) $c_i = a_i$ for all i and $\beta \in G_A$, in which case it equals

$$(-1)^{|b| \sum_{j>i} \|a_j\|} K(\mathfrak{m}_{k+1,\beta}^A(a_1 \dots a_i, a, \dots, a_k) \otimes b),$$

- (b) $c_i = b_i$ for all i and $\beta \in G_B$, in which case it equals

$$(-1)^{|a|(\sum_{j \leq i} \|b_j\| + 1)} K(a \otimes \mathfrak{m}_{k+1,\beta}^B(b_1 \dots b_i, b, \dots, b_k)).$$

We are now ready to state the main theorem of this section.

Theorem 5.3. Let (A, \mathfrak{m}^A) and (B, \mathfrak{m}^B) be commuting subalgebras of (C, μ) and equip $A \otimes B$ with the differential $\delta = \mathfrak{m}_{1,0}^A \otimes id + id \otimes \mathfrak{m}_{1,0}^B$. If $K : A \otimes B \rightarrow C$ is an injective map which induces an isomorphism on $\delta - \mu_{1,0}$ -cohomology then there is a (strict) quasi-isomorphism

$$A \otimes_\infty B \simeq C.$$

We will prove this theorem in three steps. First we will replace (C, μ) by a quasi-isomorphic A_∞ -algebra $(A \otimes B, \eta)$ whose underlying vector space is $A \otimes B$. Secondly we will construct a new filtered dg-algebra $End_{A,B}$ quasi-isomorphic to $A \otimes_\infty B$ and finally we will construct a quasi-isomorphism from $(A \otimes B, \eta)$ to $End_{A,B}$.

Proposition 5.4. Let (A, \mathfrak{m}^A) , (B, \mathfrak{m}^B) and (C, μ) be as in Theorem 5.3. Then there is a filtered A_∞ -algebra $(A \otimes B, \eta)$ quasi-isomorphic to (C, μ) , such that A and B are commuting subalgebras of $(A \otimes B, \eta)$, via the inclusions $a \mapsto a \otimes e_B$ and $b \mapsto e_A \otimes b$.

PROOF. The proof is an application of the homological perturbation lemma. By assumption on the map K , we can choose a subspace $V \subset C$ such that

$$C = K(A \otimes B) \oplus V.$$

Using this decomposition we write

$$\mu_{1,0} = \begin{pmatrix} \delta & g \\ 0 & d_V \end{pmatrix}.$$

Since K is an isomorphism on cohomology, d_V must be acyclic, thus we can choose W such that

$$V = W \oplus d_V(W).$$

We define $p : C \longrightarrow A \otimes B$ and $H : C \longrightarrow C$ as

$$\begin{aligned} p(K(a \otimes b), w_0, d_V w_1) &= a \otimes b - g(w_1), \\ H(K(a \otimes b), w_0, d_V w_1) &= (0, -w_1, 0). \end{aligned}$$

We can easily check that K , p and H define homotopy data satisfying the side conditions. Therefore, applying Theorem 2.10 and Proposition 2.11 we obtain a filtered A_∞ -algebra $(A \otimes B, \eta)$ quasi-isomorphic to (C, μ) .

Next we show that A is a subalgebra of $(A \otimes B, \eta)$. Recall that

$$\eta_{k,\beta}(a_1 \otimes e, \dots, a_k \otimes e) = \sum_{T \in \Gamma_k(\beta)} \mu_{k,T}(a_1, \dots, a_k).$$

Since, by assumption $A \subseteq A \otimes B \subseteq C$ is closed under the $\mu_{k,\beta}$ operations and $H|_{A \otimes B} = 0$ we see that, if T has internal edges, then $\mu_{k,T}(a_1, \dots, a_k) = 0$. This, together with $K(a \otimes e) = a$ implies $\eta_{k,\beta}(a_1, \dots, a_k) = \mu_{k,\beta}(a_1, \dots, a_k) = \mathbf{m}_{k,\beta}^A(a_1, \dots, a_k)$, since A is a subalgebra of C . Similarly, we prove that B is a subalgebra of $(A \otimes B, \eta)$.

Next we verify that conditions (1), (2) and (3) in definition 5.2 are satisfied. The same argument implies that

$$\begin{aligned} \eta_{k,\beta}(c_1, \dots, c_k) &= \mu_{k,\beta}(c_1, \dots, c_k), \\ \eta_{k+1,\beta}(c_1, \dots, c_i, c, c_{i+1}, \dots, c_k) &= \mu_{k,\beta}(c_1, \dots, c_i, c, c_{i+1}, \dots, c_k). \end{aligned}$$

So we conclude that (A, \mathbf{m}^A) and (B, \mathbf{m}^B) are commuting subalgebras of $(A \otimes B, \eta)$. \square

Next we introduce a new filtered dg-algebra $End_{A,B}$. As a vector space, $End_{A,B}$ is the subspace of

$$\text{Hom}_{\mathbb{K}} \left(\bigoplus_{r,s \geq 0} A \otimes B \otimes A^{\otimes r} \otimes B^{\otimes s}, A \otimes B \right)$$

satisfying

$$\rho_{r,s}(\bullet; \dots, e_A, \dots) = \rho_{r,s}(\bullet; \dots, e_B, \dots) = 0.$$

An element $\rho = \{\rho_{r,s}\}_{r,s} \in End_{A,B}$ is said to have degree $k = |\rho|$ if each $\rho_{r,s}$ has degree $k - r - s$.

To define the A_∞ operations on $End_{A,B}$, we introduce the following convenient notation

$$\begin{aligned} \overline{\mathbf{m}}_{k+1,\beta}^A(a \otimes b, a_1, \dots, a_k) &= (-1)^{|b| \sum_{i=1}^k |a_i|} \mathbf{m}_{k+1}^A(a, a_1, \dots, a_k) \otimes b, \\ \overline{\mathbf{m}}_{k+1,\beta}^B(a \otimes b, b_1, \dots, b_k) &= (-1)^{|a|} a \otimes \mathbf{m}_{k+1}^B(b, b_1, \dots, b_k). \end{aligned} \quad (22)$$

For each $\beta \in G = G_A + G_B$, we define

$$\overline{\mu}_{k,\beta} = \sum_{\substack{\beta_1 \in G_A, \beta_2 \in G_B \\ \beta_1 + \beta_2 = \beta}} \overline{\mu}_{k,\beta_1 \times \beta_2}$$

where the maps $\bar{\mu}_{k,\beta_1 \times \beta_2}$ are defined as

$$\begin{aligned} (\bar{\mu}_{0,\beta_1 \times 0})_{r,s}(x \otimes y; \mathbf{a}; \mathbf{b}) &= \begin{cases} \sum_{\beta'_1 + \beta''_1 = \beta_1} (-1)^{|y| + |\mathbf{a}|} \mathbf{m}_{r+2,\beta'_1}^A(\mathbf{m}_{0,\beta''_1}^A, x, \mathbf{a}) \otimes y & , s = 0, \\ 0 & , \text{otherwise,} \end{cases} \\ (\bar{\mu}_{0,0 \times \beta_2})_{r,s}(x \otimes y; \mathbf{a}; \mathbf{b}) &= \begin{cases} \sum_{\beta'_2 + \beta''_2 = \beta_2} x \otimes \mathbf{m}_{s+2,\beta'_2}^B(\mathbf{m}_{0,\beta''_2}^B, y, \mathbf{b}) & , r = 0, \\ 0 & , \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} \bar{\mu}_{1,\beta_1 \times 0}(\rho)_{r,s}(x \otimes y; \mathbf{a}; \mathbf{b}) &= \sum (-1)^{1 + |\mathbf{a}^{(2)}| + |\mathbf{b}|} \bar{\mathbf{m}}_{r-i+1,\beta_1}^A \left(\rho_{i,s}(x \otimes y; \mathbf{a}^{(1)}; \mathbf{b}), \mathbf{a}^{(2)} \right) \\ &\quad + \sum (-1)^{|\rho|} \rho_{r-i,s} \left(\bar{\mathbf{m}}_{i+1,\beta_1}^A(x \otimes y; \mathbf{a}^{(1)}); \mathbf{a}^{(2)}; \mathbf{b} \right) \\ &\quad + \sum (-1)^{|\rho| + |x \otimes y| + |\mathbf{a}^{(1)}|} \rho_{r-j+1,s} \left(x \otimes y; \mathbf{a}^{(1)}, \mathbf{m}_{j,\beta_1}^A(\mathbf{a}^{(2)}), \mathbf{a}^{(3)}; \mathbf{b} \right), \\ \bar{\mu}_{1,0 \times \beta_2}(\rho)_{r,s}(x \otimes y; \mathbf{a}; \mathbf{b}) &= \sum -\bar{\mathbf{m}}_{j+1,\beta_2}^B \left(\rho_{r,s-j}(x \otimes y; \mathbf{a}; \mathbf{b}^{(1)}), \mathbf{b}^{(2)} \right) \\ &\quad + \sum (-1)^{|\rho| + |\mathbf{a}| + |\mathbf{b}^{(1)}|} \rho_{r,s-j} \left(\bar{\mathbf{m}}_{j+1,\beta_2}^B(x \otimes y; \mathbf{b}^{(1)}); \mathbf{a}; \mathbf{b}^{(2)} \right) \\ &\quad + \sum (-1)^{|\rho| + |x \otimes y| + |\mathbf{a}| + |\mathbf{b}^{(1)}|} \rho_{r,s-j+1} \left(x \otimes y; \mathbf{a}; \mathbf{b}^{(1)}, \mathbf{m}_{j,\beta_2}^B(\mathbf{b}^{(2)}), \mathbf{b}^{(3)} \right). \\ \bar{\mu}_{2,0}(\rho, \tau)(x \otimes y; \mathbf{a}; \mathbf{b}) &= \sum (-1)^{|\rho| + |\mathbf{b}^{(1)}| + |\mathbf{a}^{(2)}|} \rho_{r-i,s-j} \left(\tau_{i,j}(x \otimes y; \mathbf{a}^{(1)}; \mathbf{b}^{(1)}); \mathbf{a}^{(2)}; \mathbf{b}^{(2)} \right) \end{aligned}$$

and all other $\bar{\mu}_{k,\beta_1 \times \beta_2} = 0$. We have the following

Proposition 5.5. *$End_{A,B} = (End_{A,B}, \bar{\mu})$ is a filtered dg-algebra.*

PROOF. The proof is straightforward computation combining the proofs of Lemmas 3.1 and 3.5. We simply highlight the main points.

Just as in Lemma 3.1 we can show that

$$\sum_{\beta'_1 + \beta''_1 = \beta_1} \bar{\mu}_{1,\beta'_1 \times 0}(\bar{\mu}_{0,\beta''_1 \times 0}) = 0 = \sum_{\beta'_2 + \beta''_2 = \beta_2} \bar{\mu}_{1,0 \times \beta'_2}(\bar{\mu}_{0,0 \times \beta''_2}).$$

This, combined with the equalities

$$\bar{\mu}_{1,\beta_1 \times 0}(\bar{\mu}_{0,0 \times \beta_2}) = \bar{\mu}_{1,0 \times \beta_2}(\bar{\mu}_{0,\beta_1 \times 0}) = 0,$$

immediately implies the A_∞ -equation with no inputs. For the next equation, we note that

$$\bar{\mu}_{1,\beta_1 \times 0}(\bar{\mu}_{1,0 \times \beta_2}(\rho)) + \bar{\mu}_{1,0 \times \beta_2}(\bar{\mu}_{1,\beta_1 \times 0}(\rho)) = 0.$$

and as in Lemma 3.1 we can show

$$\sum_{\beta'_1 + \beta''_1 = \beta_1} \bar{\mu}_{1,\beta'_1 \times 0}(\bar{\mu}_{1,\beta''_1 \times 0}(\rho)) + \bar{\mu}_{2,0}(\bar{\mu}_{0,\beta_1 \times 0}, \rho) + (-1)^{|\rho|} \bar{\mu}_{2,0}(\rho, \bar{\mu}_{0,\beta_1 \times 0}) = 0$$

and the analogous equation for $0 \times \beta_2$. These combine to prove the A_∞ equation with one input. Finally, we can prove

$$\bar{\mu}_{1,\gamma}(\bar{\mu}_{2,0}(\rho, \tau)) + \bar{\mu}_{2,0}(\bar{\mu}_{1,\gamma}(\rho), \tau) + (-1)^{|\rho|} \bar{\mu}_{2,0}(\rho, \bar{\mu}_{1,\gamma}(\tau)) = 0$$

for $\gamma = \beta_1 \times 0$ or $\gamma = 0 \times \beta_2$, and

$$\bar{\mu}_{2,0}(\bar{\mu}_{2,0}(\rho, \tau), \eta) + (-1)^{|\rho|} \bar{\mu}_{2,0}(\rho, \bar{\mu}_{2,0}(\tau, \eta)) = 0,$$

by straightforward computations. This finishes the proof of the A_∞ equations, to complete the proof we just note that

$$(id)_{r,s}(x \otimes y; \mathbf{a}; \mathbf{b}) = \begin{cases} x \otimes y & , r = s = 0, \\ 0 & , \text{otherwise,} \end{cases}$$

is a unit for $End_{A,B}$.

□

Proposition 5.6. *Let A and B be filtered A_∞ -algebras, the map*

$$S(\rho \otimes \tau)(x \otimes y; \mathbf{a}; \mathbf{b}) = (-1)^{|y||\mathbf{a}| + |\tau|(|x| + |\mathbf{a}|)} \rho(x; \mathbf{a}) \otimes \tau(y; \mathbf{b}),$$

defines a naive quasi-isomorphism $S : End_A \otimes_{dg} End_B \longrightarrow End_{A,B}$.

PROOF. It is straightforward check that

$$\begin{aligned} S(\mu_{0,\gamma}^\otimes) &= \bar{\mu}_{0,\gamma}, \\ S(\mu_{1,\gamma}^\otimes(\varphi)) &= \bar{\mu}_{1,\gamma}(S(\varphi)) \end{aligned}$$

and

$$S(\mu_{2,\gamma}^\otimes(\varphi_1, \varphi_2)) = \bar{\mu}_{2,0}(S(\varphi_1), S(\varphi_2)),$$

for any $\varphi = \rho \otimes \tau$ and $\gamma = \beta_1 \times 0$ or $\gamma = 0 \times \beta_2$. This shows that S is a naive A_∞ -homomorphism. We need to check that it induces an isomorphism in cohomology

$$S_* : H^*(End_A \otimes_{dg} End_B, \mu_{1,0}^\otimes) \longrightarrow H^*(End_{A,B}, \bar{\mu}_{1,0}).$$

In fact, when A and B are finite dimensional vector spaces S is an isomorphism. In the general case we argue as follows. Denote by I the following composition

$$A \otimes B \xrightarrow{F_{1,0}^A \otimes F_{1,0}^B} End_A \otimes End_B \xrightarrow{S} End_{A,B}.$$

Lemma 3.3 states that $F_{1,0}^A$ and $F_{1,0}^B$ induce isomorphisms in cohomology, therefore by the Künneth theorem, so does $F_{1,0}^A \otimes F_{1,0}^B$. We have reduced the proof to showing that I induces an isomorphism in cohomology. We will do this by providing a homotopy inverse. Define $\bar{P} : End_{A,B} \longrightarrow A \otimes B$ as

$$\bar{P}(\rho) = (-1)^{|\rho|} \rho_{0,0}(e_A \otimes e_B).$$

From the definition

$$I(u \otimes v)(x \otimes y; \mathbf{a}; \mathbf{b}) = (-1)^{|y||\mathbf{a}| + (|x| + |\mathbf{a}|)|v|} \mathbf{m}_{r+2,0}^A(u, x, \mathbf{a}) \otimes \mathbf{m}_{s+2,0}^B(v, y, \mathbf{b}),$$

which readily implies $\bar{P} \circ I = id_{A \otimes B}$. The composition $I \circ \bar{P}$ is homotopic to the identity in $End_{A,B}$. In fact, if we define

$$(K\rho)_{r,s}(x \otimes y; \mathbf{a}; \mathbf{b}) = (-1)^{\xi_1} \bar{\mathbf{m}}_{r+2,0}^A(\rho_{0,s+1}(E_a \otimes e_B; y; \mathbf{b}), x, \mathbf{a}) + (-1)^{\xi_2} \rho_{r+1,s}(e_A \otimes y; x, \mathbf{a}; \mathbf{b}),$$

where $\xi_1 = |\mathbf{b}|(|\mathbf{a}| + |x|) + |x||y| + |y|$ and $\xi_2 = |\rho| + |y||x|$, we can see that

$$K\bar{\mu}_{1,0} + \bar{\mu}_{1,0}K = I \circ \bar{P} - id_{End_{A,B}}.$$

This is a simple albeit long computation, entirely analogous to Lemma 3.3, so we will omit it. □

The third and main step in the proof of Theorem 5.3 is the construction of an A_∞ -homomorphism

$$F : (A \otimes B, \eta) \longrightarrow End_{A,B}.$$

For this, we need to introduce the notion of shuffle product.

Definition 5.7. Let $r, s \geq 0$ be integers. We say a permutation $\sigma \in S_{r+s}$ is a (r, s) -*shuffle* if $\sigma(i) < \sigma(j)$ for $i < j \leq r$ or $r+1 \leq i < j$.

Consider $\mathbf{a} = a_1 \otimes \dots \otimes a_r \in A^{\otimes r}$ and $\mathbf{b} = b_1 \otimes \dots \otimes b_s \in B^{\otimes s}$ and denote

$$u_i = \begin{cases} a_i & , 1 \leq i \leq r, \\ b_{i-r} & , r < i \leq r+s. \end{cases}$$

Given a (r, s) -shuffle σ , we define

$$\sigma(\mathbf{a}, \mathbf{b}) = (-1)^\epsilon u_{\sigma^{-1}(1)} \otimes \dots \otimes u_{\sigma^{-1}(r+s)},$$

with

$$\epsilon = \sum_{\substack{i \leq r < j \\ \sigma(j) < \sigma(i)}} \|u_j\| \|u_i\|.$$

With the same notation, we define

$$Sh(\mathbf{a}; \mathbf{b}) = \sum_{\sigma} \sigma(\mathbf{a}; \mathbf{b})$$

where we sum over all (r, s) -shuffles σ . In the next lemma we establish some properties of the shuffle Sh that we will need later.

Lemma 5.8. *Let A and B be commuting subalgebras of $(A \otimes B, \eta_{k,\beta})$ as in Proposition 5.4. We have the following*

$$(1) \quad \sum_{(Sh(\mathbf{a}; \mathbf{b}))} Sh(\mathbf{a}; \mathbf{b})^{(1)} \otimes Sh(\mathbf{a}; \mathbf{b})^{(2)} = \sum_{(\mathbf{a})(\mathbf{b})} (-1)^{\|\mathbf{a}^{(2)}\| \|\mathbf{b}^{(1)}\|} Sh(\mathbf{a}^{(1)}; \mathbf{b}^{(1)}) \otimes Sh(\mathbf{a}^{(2)}; \mathbf{b}^{(2)}),$$

$$(2) \quad Sh(\mathbf{a}_1, x, \mathbf{a}_2; \mathbf{b}) = \sum_{(\mathbf{b})} (-1)^{\|\mathbf{b}^{(1)}\|(\|x\| + \|\mathbf{a}^{(2)}\|)} Sh(\mathbf{a}_1; \mathbf{b}^{(1)}) \otimes x \otimes Sh(\mathbf{a}_2; \mathbf{b}^{(2)}),$$

$$\text{and} \quad Sh(\mathbf{a}; \mathbf{b}_1, y, \mathbf{b}_2) = \sum_{(\mathbf{a})} (-1)^{\|\mathbf{a}^{(2)}\|(\|\mathbf{b}^{(1)}\| + \|y\|)} Sh(\mathbf{a}^{(1)}; \mathbf{b}_1) \otimes y \otimes Sh(\mathbf{a}^{(2)}; \mathbf{b}_2),$$

$$(3) \quad \eta_{r+s+1, \beta}(z, Sh(\mathbf{a}; \mathbf{b})) = \begin{cases} \overline{\mathbf{m}}_{r+1, \beta \times 0}^A(z, \mathbf{a}) & , \beta \in G_A, s = 0, \\ \overline{\mathbf{m}}_{s+1, 0 \times \beta}^B(z, \mathbf{b}) & , \beta \in G_B, r = 0, \\ \overline{\mathbf{m}}_{1, \beta \times 0}^A(z) + \overline{\mathbf{m}}_{1, 0 \times \beta}^B(z) & , r, s = 0, \\ 0 & , \text{otherwise}, \end{cases}$$

$$(4) \quad \eta_{r+s, \beta}(Sh(\mathbf{a}; \mathbf{b})) = \begin{cases} \mathbf{m}_{r, \beta}^A(\mathbf{a}) & , \beta \in G_A, s = 0, r > 0, \\ \mathbf{m}_{s, \beta}^B(\mathbf{b}) & , \beta \in G_B, r = 0, s > 0, \\ \mathbf{m}_{0, \beta}^A(z) + \mathbf{m}_{0, \beta}^B(z) & , r, s = 0, \\ 0 & , \text{otherwise}, \end{cases}$$

PROOF.

1. This is part of the statement that, for any vector space V , the tensor algebra $TV = \oplus_{k \geq 0} V^{\otimes k}$, equipped with $\Delta(v) = \sum_{(v)} v^{(1)} \otimes v^{(2)}$ and Sh is a bialgebra. See [6] for a proof of this fact.
2. Straightforward check, left to the reader.
3. By definition, we have

$$\eta_{r+s+1, \beta}(z, Sh(\mathbf{a}; \mathbf{b})) = \sum_{\sigma} \eta_{r+s+1, \beta}(z, \sigma(\mathbf{a}; \mathbf{b})).$$

Since A and B are commuting subalgebras,

$$\eta_{r+s+1,\beta}(z, \sigma(\mathbf{a}; \mathbf{b})) = 0$$

unless one of the following happens: $s = 0$ and $\beta \in G_A$, in which case $\sigma = id$ and $\eta_{r+1,\beta}(z, \mathbf{a}) = \overline{\mathbf{m}}_{r+1,\beta \times 0}^A(z, \mathbf{a})$; or $r = 0$ and $\beta \in G_B$ which implies $\eta_{r+1,\beta}(z, \mathbf{b}) = \overline{\mathbf{m}}_{s+1,0 \times \beta}^B(z, \mathbf{b})$. When $r = s = 0$ we sum the two contributions and get $\mathbf{m}_{1,\beta \times 0}^A(z) + \overline{\mathbf{m}}_{1,0 \times \beta}^B(z)$.

4. This is the same as (3), with one extra case: $\mathbf{a} = a_1$, $\mathbf{b} = b_1$ and $\beta = 0$. In this case

$$\eta_{2,0}(Sh(\mathbf{a}; \mathbf{b})) = \eta_{2,0}(a_1, b_1) + (-1)^{\|b_1\| \|a_1\|} \eta_{2,0}(b_1, a_1) = 0,$$

since A and B commute. □

Proposition 5.9. *Let A and B be commuting subalgebras of $(A \otimes B, \eta)$, as in Proposition 5.4. Consider the sequence of maps $F_{k,\beta} : (A \otimes B)^{\otimes k} \rightarrow \text{End}_{A,B}$ defined as*

$$F_{k,\beta}(c_1, \dots, c_k)(z; \mathbf{a}; \mathbf{b}) = \eta_{k+r+s+1,\beta}(c_1, \dots, c_k, z, Sh(\mathbf{a}; \mathbf{b}))$$

for $c_i, z = x \otimes y \in A \otimes B$ and $k > 0$. These define a strict, filtered A_∞ -homomorphism.

PROOF. We first check the A_∞ equation, with no inputs:

$$\overline{\mu}_{0,\beta} = \sum_{\beta' + \beta'' = \beta} F_{1,\beta'}(\eta_{0,\beta''}).$$

Since A and B are commuting subalgebras, $\eta_{0,\beta''} = \mathbf{m}_{0,\beta''}^A + \mathbf{m}_{0,\beta''}^B$, so we compute

$$\begin{aligned} \sum_{\beta' + \beta'' = \beta} F_{1,\beta'}(\eta_{0,\beta''})(z; \mathbf{a}; \mathbf{b}) &= \sum_{\beta' + \beta'' = \beta} F_{1,\beta'}(\mathbf{m}_{0,\beta''}^A + \mathbf{m}_{0,\beta''}^B)(z; \mathbf{a}; \mathbf{b}) \\ &= \sum_{\beta' + \beta'' = \beta} \eta_{r+s+2,\beta'}(\mathbf{m}_{0,\beta''}^A, z, Sh(\mathbf{a}; \mathbf{b})) + \eta_{r+s+2,\beta'}(\mathbf{m}_{0,\beta''}^B, z, Sh(\mathbf{a}; \mathbf{b})). \end{aligned}$$

Again, by commutativity of A and B this equals

$$\sum_{\beta' + \beta'' = \beta} (-1)^{\|y\| \|\mathbf{a}\|} \mathbf{m}_{r+2,\beta'}^A(\mathbf{m}_{0,\beta''}^A, x, \mathbf{a}) \otimes y + x \otimes \mathbf{m}_{s+2,\beta'}^B(\mathbf{m}_{0,\beta''}^B, y, \mathbf{b})$$

with the convention that the first (respectively second) term is zero if $s \neq 0$ (respectively $r \neq 0$). This is exactly the definition of $\overline{\mu}_{0,\beta}(z; \mathbf{a}; \mathbf{b})$.

To prove the A_∞ -homomorphism equation with $k > 0$ inputs we consider the A_∞ -equation on $A \otimes B$, with inputs $\mathbf{c} = c_1 \otimes \dots \otimes c_k$, z and $\sigma(\mathbf{a}; \mathbf{b})$ for some shuffle σ . Summing over σ , we obtain the equation

$$\begin{aligned} 0 &= \sum (-1)^{\|\mathbf{c}^{(1)}\|} \eta_{j',\beta'}(\mathbf{c}^{(1)}, \eta_{j'',\beta''}(\mathbf{c}^{(2)}), \mathbf{c}^{(3)}, z, Sh(\mathbf{a}; \mathbf{b})) + \\ &+ \sum (-1)^{\|\mathbf{c}^{(1)}\|} \eta_{j',\beta'}(\mathbf{c}^{(1)}, \eta_{j'',\beta''}(\mathbf{c}^{(2)}, z, Sh(\mathbf{a}; \mathbf{b})^{(1)}), Sh(\mathbf{a}; \mathbf{b})^{(2)}) + \\ &+ \sum (-1)^{\|\mathbf{c}\| + \|z\| + \|Sh(\mathbf{a}; \mathbf{b})^{(1)}\|} \eta_{j',\beta'}(\mathbf{c}, z, Sh(\mathbf{a}; \mathbf{b})^{(1)}, \eta_{j'',\beta''}(Sh(\mathbf{a}; \mathbf{b})^{(2)}), Sh(\mathbf{a}; \mathbf{b})^{(3)}). \end{aligned} \quad (23)$$

By definition, the first sum in (23) equals

$$\sum (-1)^{\|\mathbf{c}^{(1)}\|} F_{k-j+1,\beta'}(\mathbf{c}^{(1)}, \eta_{j,\beta''}(\mathbf{c}^{(2)}), \mathbf{c}^{(3)})(z; \mathbf{a}; \mathbf{b}). \quad (24)$$

Applying part (1) of Lemma 5.8 to the second sum in (23) we see it equals

$$\sum_{(\mathbf{c}), (\mathbf{a}), (\mathbf{b})} (-1)^{\|\mathbf{c}^{(1)}\| + \|\mathbf{b}^{(1)}\| + \|\mathbf{a}^{(2)}\|} \eta_{j', \beta'} \left(\mathbf{c}^{(1)}, \eta_{j'', \beta''} \left(\mathbf{c}^{(2)}, z, Sh \left(\mathbf{a}^{(1)}; \mathbf{b}^{(1)} \right) \right), Sh \left(\mathbf{a}^{(2)}; \mathbf{b}^{(2)} \right) \right)$$

We will further decompose this sum into two parts. The first consists of the terms where $\mathbf{c}^{(1)}, \mathbf{c}^{(2)} \neq \mathbf{c}$. It equals

$$\begin{aligned} & \sum_{\mathbf{c}^{(1)}, \mathbf{c}^{(2)} \neq \mathbf{c}} (-1)^{\|\mathbf{c}^{(1)}\| + \|\mathbf{b}^{(1)}\| + \|\mathbf{a}^{(2)}\|} \eta_{j', \beta'} \left(\mathbf{c}^{(1)}, \eta_{j'', \beta''} \left(\mathbf{c}^{(2)}, z, Sh \left(\mathbf{a}^{(1)}; \mathbf{b}^{(1)} \right) \right), Sh \left(\mathbf{a}^{(2)}; \mathbf{b}^{(2)} \right) \right) \\ &= - \sum \bar{\mu}_{2,0} \left(F_{k', \beta'} \left(\mathbf{c}^{(1)} \right), F_{k'', \beta''} \left(\mathbf{c}^{(2)} \right) \right) (z; \mathbf{a}; \mathbf{b}), \end{aligned} \quad (25)$$

because $|F_{k, \beta}(\mathbf{c})| = \|\mathbf{c}\| + 1$. The second consists of the terms where $\mathbf{c}^{(1)} = \mathbf{c}$ or $\mathbf{c}^{(2)} = \mathbf{c}$ and so it is equal to

$$\begin{aligned} & \sum (-1)^{\|\mathbf{b}^{(1)}\| + \|\mathbf{a}^{(2)}\|} \eta_{j', \beta'} \left(\eta_{j'', \beta''} \left(\mathbf{c}, z, Sh \left(\mathbf{a}^{(1)}; \mathbf{b}^{(1)} \right) \right), Sh \left(\mathbf{a}^{(2)}; \mathbf{b}^{(2)} \right) \right) \\ &+ \sum (-1)^{\|\mathbf{c}^{(1)}\| + \|\mathbf{b}^{(1)}\| + \|\mathbf{a}^{(2)}\|} \eta_{j', \beta'} \left(\mathbf{c}, \eta_{j'', \beta''} \left(z, Sh \left(\mathbf{a}^{(1)}; \mathbf{b}^{(1)} \right) \right), Sh \left(\mathbf{a}^{(2)}; \mathbf{b}^{(2)} \right) \right) \\ &= \sum_{\beta' \in G_A} (-1)^{\|\mathbf{b}\| + \|\mathbf{a}^{(2)}\|} \bar{\mathbf{m}}_{j', \beta' \times 0}^A \left(\eta_{j'', \beta''} \left(\mathbf{c}, z, Sh \left(\mathbf{a}^{(1)}; \mathbf{b} \right) \right), \mathbf{a}^{(2)} \right) \\ &+ \sum_{\beta' \in G_B} \bar{\mathbf{m}}_{j', 0 \times \beta'}^B \left(\eta_{j'', \beta''} \left(\mathbf{c}, z, Sh \left(\mathbf{a}; \mathbf{b}^{(1)} \right) \right), \mathbf{b}^{(2)} \right) \\ &+ \sum_{\beta'' \in G_A} (-1)^{\|\mathbf{c}\|} \eta_{j', \beta'} \left(\mathbf{c}, \bar{\mathbf{m}}_{j'', \beta'' \times 0}^A \left(z, \mathbf{a}^{(1)} \right), Sh \left(\mathbf{a}^{(2)}; \mathbf{b} \right) \right) \\ &+ \sum_{\beta'' \in G_B} (-1)^{\|\mathbf{c}\| + \|\mathbf{b}^{(1)}\| + \|\mathbf{a}\|} \eta_{j', \beta'} \left(\mathbf{c}, \bar{\mathbf{m}}_{j'', 0 \times \beta''}^B \left(z, \mathbf{b}^{(1)} \right), Sh \left(\mathbf{a}; \mathbf{b}^{(2)} \right) \right), \end{aligned} \quad (26)$$

by part (3) of Lemma 5.8. Finally, we apply part (1) of Lemma 5.8 to the third sum in (23) and we get

$$\sum (-1)^\nu \eta_{j', \beta'} \left(\mathbf{c}, z, Sh \left(\mathbf{a}^{(1)}; \mathbf{b}^{(1)} \right), \eta_{j'', \beta''} \left(Sh \left(\mathbf{a}^{(2)}; \mathbf{b}^{(2)} \right) \right), Sh \left(\mathbf{a}^{(3)}; \mathbf{b}^{(3)} \right) \right)$$

where $\nu = \|\mathbf{c}\| + \|z\| + \|Sh(\mathbf{a}^{(1)}; \mathbf{b}^{(1)})\| + \|\mathbf{b}^{(1)}\| (\|\mathbf{a}^{(2)}\| + \|\mathbf{a}^{(3)}\|) + \|\mathbf{b}^{(2)}\| \|\mathbf{a}^{(3)}\|$. Observe that in this formula $\eta_{j'', \beta''} (Sh(\mathbf{a}^{(2)}; \mathbf{b}^{(2)}))$ equals $\mathbf{m}_{j'', \beta''}^A(\mathbf{a}^{(2)})$ or $\mathbf{m}_{j'', \beta''}^B(\mathbf{b}^{(2)})$, by part (4) of Lemma 5.8. Then, applying (2) of Lemma 5.8, with $x = \mathbf{m}_{j'', \beta''}^A(\mathbf{a}^{(2)})$ and $y = \mathbf{m}_{j'', \beta''}^B(\mathbf{b}^{(2)})$ we conclude the above formula equals

$$\begin{aligned} & \sum_{\beta'' \in G_A} (-1)^{\|\mathbf{c}\| + \|z\| + \|\mathbf{a}^{(1)}\|} \eta_{j', \beta'} \left(\mathbf{c}, z, Sh \left(\mathbf{a}^{(1)}, \mathbf{m}_{\beta''}^A(\mathbf{a}^{(2)}), \mathbf{a}^{(3)}; \mathbf{b} \right) \right) \\ &+ \sum_{\beta'' \in G_B} (-1)^{\|\mathbf{c}\| + \|z\| + \|\mathbf{a}\| + \|\mathbf{b}^{(1)}\|} \eta_{j', \beta'} \left(\mathbf{c}, z, Sh \left(\mathbf{a}, \mathbf{b}^{(1)}, \mathbf{m}_{\beta''}^B(\mathbf{b}^{(2)}), \mathbf{b}^{(3)} \right) \right). \end{aligned} \quad (27)$$

Comparing with the definition of $\bar{\mu}_{1, \beta}$, we see that

$$(26) + (27) = - \sum \bar{\mu}_{1, \beta'} (F_{k, \beta''}(\mathbf{c})) (z; \mathbf{a}; \mathbf{b}). \quad (28)$$

Assembling (24), (25) and (28) we conclude that (23) is equivalent to

$$\begin{aligned} & \sum (-1)^{\|\mathbf{c}^{(1)}\|} F_{k-j+1, \beta'} \left(\mathbf{c}^{(1)}, \eta_{j, \beta''} \left(\mathbf{c}^{(2)} \right), \mathbf{c}^{(3)} \right) (z; \mathbf{a}; \mathbf{b}) \\ &- \sum \bar{\mu}_{2,0} \left(F_{k', \beta'} \left(\mathbf{c}^{(1)} \right), F_{k'', \beta''} \left(\mathbf{c}^{(2)} \right) \right) (z; \mathbf{a}; \mathbf{b}) \\ &- \sum \bar{\mu}_{1, \beta'} (F_{k, \beta''}(\mathbf{c})) (z; \mathbf{a}; \mathbf{b}) = 0. \end{aligned}$$

This completes the proof that $F_{k,\beta}$ defines an A_∞ -homomorphism. Lastly, the equality

$$\begin{aligned} F_{k,\beta}(\mathbf{c}_1, e_A \otimes e_B, \mathbf{c}_2)(z; \mathbf{a}; \mathbf{b}) &= \\ &= \eta_{k+r+s+1,\beta}(\mathbf{c}_1, e_A \otimes e_B, \mathbf{c}_2, z, Sh(\mathbf{a}; \mathbf{b})) = \begin{cases} z & , (k, \beta) = (1, 0), r = s = 0, \\ 0 & , \text{otherwise,} \end{cases} \end{aligned}$$

shows that $F_{k,\beta}$ is unital. \square

Proof of Theorem 5.3: Assembling Propositions 5.4, 5.6 and 5.9 we have the sequence of A_∞ -homomorphisms

$$(C, \mu) \xrightarrow{\varphi} (A \otimes B, \eta) \xrightarrow{F} End_{A,B} \xleftarrow{S} A \otimes_\infty B.$$

We already saw that φ and S are quasi-isomorphisms, so all that is left is to show that F is also a quasi-isomorphism. Recall the map \overline{P} introduced in the proof of Proposition 5.6. It satisfies

$$\overline{P}(F_{1,0}(u \otimes v)) = (-1)^{(|u|+|v|)} \eta_{2,0}(u \otimes v, e_A \otimes e_B) = u \otimes v,$$

that is $\overline{P} \circ F_{1,0} = id_{A \otimes B}$. But we already saw that \overline{P} is an isomorphism on cohomology, so we conclude that $F_{1,0}$ induces an isomorphism in cohomology. This completes the proof of the theorem. \square

6 Bounding cochains on the tensor product

6.1 Bounding Cochains

In this subsection we will describe (some of) the bounding cochains on $A \otimes_\infty B$ using the description of the tensor product given by Theorem 4.1. Namely we will construct a map

$$\boxtimes : MC(A) \times MC(B) \longrightarrow MC(A \otimes B, \mathfrak{m}^\otimes).$$

We start with the following preliminary

Lemma 6.1. *Consider a tree $U \in L_n$, $\beta \in G_A$ and $a_1, \dots, a_n \in A$. Let j_a be the number of a_i such that $a_i = e_A$ and denote by $b(U)$ the number of vertices of U of valency 2. If $U_\beta^A(a_1, \dots, a_n) \neq 0$, then $b(U) \geq j_a$ unless $j_a = n$ and $\beta = 0$, in which case $b(U) = n - 1$. Moreover the same is true if $U \in R_n$.*

PROOF. Let $w_1^i, \dots, w_{l_i}^i$ be the set of internal vertices of P_i , the path from the i -th leaf of U to the root (w_1^i adjacent to the leaf, $w_{l_i}^i$ adjacent to the root). If $a_i = e_A$ then w_1^i must be a binary vertex (of weight zero) since A is unital. This would imply $b(U) \geq j_a$ unless there is $1 \leq l \leq n$ such that $a_l = e_A$ and $a_{l+1} = e_A$. In that case $\mathfrak{m}_{2,0}(a_l, a_{l+1}) = e_A$ and so w_2^l must also be binary.

By definition of L_n , the edge from w_1^l to w_2^l must be right leaning and the other (since w_2^l has valency two) incoming edge at w_2^l must be adjacent to a leaf. This implies $b(U) \geq j_a$ unless $a_{l+2} = e_A$. Iterating this argument we conclude that the only way $b(U) < j_a$ is if $l = 1$ and $a_i = e_A$ for all i . This implies U is the minimal binary tree, $j_a = n$ and $b(U) = n - 1$.

The same argument (interchanging right and left) applied to $U \in R_n$, proves the second claim. The only difference is that now when $j_a = n$, U is the maximal binary tree. \square

Proposition 6.2. *Let $x \in \widehat{MC}(A)$ and $y \in \widehat{MC}(B)$ and define $x \boxtimes y = x \otimes e_B + e_A \otimes y \in \widehat{A \otimes B}_0$. Then $x \boxtimes y \in \widehat{MC}(A \otimes B, \mathfrak{m}^\otimes)$ and*

$$\mathcal{P}(x \boxtimes y) = \mathcal{P}(x) + \mathcal{P}(y).$$

PROOF. From Proposition 4.3 we have

$$\mathfrak{m}_{n, \beta_1 \times \beta_2}^\otimes(x \boxtimes y, \dots, x \boxtimes y) = \sum_{\substack{U, W \\ a_i, b_j}} \pm U_{\beta_1}^A(a_1, \dots, a_n) \otimes W_{\beta_2}^B(b_1, \dots, b_n)$$

where for each i , $a_i = x$ and $b_i = e_B$, or $a_i = e_A$ and $b_i = y$. Moreover $U \in L_n$, $W \in R_n$ and $|E_{int}(U)| + |E_{int}(W)| = n - 2$. Equivalently $|V(U)| + |V(W)| = n$ and therefore

$$b(U) + b(W) \leq n \quad (29)$$

Observe that $j_a + j_b = n$, then if $j_a, j_b \neq 0$,

$$n = j_a + j_b \leq b(U) + b(W) \leq n$$

by the previous lemma. This implies that both U and W are binary and therefore $|V(U)| = n - 1 = |V(W)|$ which contradicts (29), unless $n = 2$. Therefore either $n = 2$, or $j_a = n$ and $\beta_1 = 0$, or $j_b = n$ and $\beta_2 = 0$. When $n \neq 2$, U and W are (respectively) the minimal and maximal binary trees and we get

$$\mathfrak{m}_{n, \beta_1 \times \beta_2}^\otimes(x \boxtimes y, \dots, x \boxtimes y) = \begin{cases} \mathfrak{m}_{n, \beta_1}(x, \dots, x) \otimes e_B & , \beta_2 = 0, \\ e_A \otimes \mathfrak{m}_{n, \beta_2}(y, \dots, y) & , \beta_1 = 0, \\ 0 & , \beta_1, \beta_2 \neq 0. \end{cases}$$

When $n = 2$ there are two extra terms,

$$\mathfrak{m}_{2, \beta_1 \times \beta_2}^\otimes(x \otimes e_B, e_A \otimes y) + \mathfrak{m}_{2, \beta_1 \times \beta_2}^\otimes(e_A \otimes y, x \otimes e_B) = 0$$

which cancel because A and B are unital. Therefore we have

$$\begin{aligned} \sum_n \mathfrak{m}_n^\otimes(x \boxtimes y, \dots, x \boxtimes y) &= \left[\sum_n \mathfrak{m}_n(x, \dots, x) \right] \otimes e_B + e_A \otimes \left[\sum_n \mathfrak{m}_n(y, \dots, y) \right] \\ &= \mathcal{P}(x) e_A \otimes e_B + e_A \otimes \mathcal{P}(y) e_B \\ &= (\mathcal{P}(x) + \mathcal{P}(y)) e_{A \otimes B}, \end{aligned} \quad (30)$$

because x and y are bounding cochains. \square

Next we will show that the map \boxtimes preserves gauge equivalence. But for this we need to make a small digression on models for $A^{[0,1]}$. As in [14], we will construct a model for $A^{[0,1]}$ as the tensor product of A with a filtered dg-algebra I . The dg-algebra I is the (normalized) cochain complex of the standard one-simplex.

Consider the graded vector space $I = I^0 \oplus I^1$, with I^0 generated by elements u_0 and u_1 and I^1 generated by h . We define operations $\mu_{k,\beta}$ on I by setting

$$\begin{aligned} -\mu_{1,0}(u_0) &= h = \mu_{1,0}(u_1), \\ \mu_{2,0}(u_0, h) &= h = -\mu_{2,0}(h, u_1), \quad \mu_{2,0}(u_0, u_0) = u_0, \quad \mu_{2,0}(u_1, u_1) = u_1, \end{aligned}$$

and defining all other operations to be trivial.

Lemma 6.3. *(I, μ) is a $\{0\}$ -gapped filtered dg-algebra with unit $e = u_0 + u_1$. There are naive quasi-isomorphisms $\bar{i} : \mathbb{K} \rightarrow I$ and $\bar{p}_0, \bar{p}_1 : I \rightarrow \mathbb{K}$ satisfying $\bar{p}_0 \circ \bar{i} = \bar{p}_1 \circ \bar{i} = id_{\mathbb{K}}$. Additionally the map $\bar{p}_0 \oplus \bar{p}_1 : I \rightarrow \mathbb{K} \oplus \mathbb{K}$ is a surjection.*

PROOF. This is a simple computation. The map \bar{i} is defined as $\bar{i}(1) = u_0 + u_1$ and \bar{p}_j is defined as the projection to the subspace generated by u_j . \square

Proposition 6.4. *The filtered A_∞ -algebra $(A \otimes I, \mathfrak{m}^\otimes)$ together with the naive quasi-isomorphisms $p_j = id \otimes \bar{p}_j : A \otimes I \longrightarrow A \otimes \mathbb{K} = A$ and $i = id \otimes \bar{i} : A = A \otimes \mathbb{K} \longrightarrow A \otimes I$ is a model for $A^{[0,1]}$.*

PROOF. We just need to observe that given naive maps $f : A_1 \longrightarrow A_2$ and $g : B_1 \longrightarrow B_2$, the naive map $f \otimes g : A_1 \otimes B_1 \longrightarrow A_2 \otimes B_2$ is an A_∞ -homomorphism. All the properties are then straightforward. \square

Remark 6.5. *This model for $A^{[0,1]}$ is exactly the one described in [5, Lemma 4.2.25]. The other explicit model for $A^{[0,1]}$ given in [5, Lemma 4.2.13] can also be seen to be a tensor product. In fact, $Map([0,1], A)$ is the tensor product of A with the de Rham complex of differential forms on the unit interval $[0,1]$.*

Using this model for $A^{[0,1]}$ we can describe explicitly gauge equivalence. Two bounding cochains x_0 and x_1 are gauge equivalent if $\mathcal{P}(x_0) = \mathcal{P}(x_1)$ and there is $c = \sum c_i T^{\lambda_i}$ with each c_i of even degree (or degree zero in the graded case) satisfying

$$x_0 - x_1 = \sum_{i,j} \mathfrak{m}_{i+j+1}(x_0, \dots, x_0, c, x_1, \dots, x_1). \quad (31)$$

After this digression we are now ready to prove the following

Proposition 6.6. *The map \boxtimes preserves gauge equivalence and so it descends to a map*

$$\boxtimes : MC(A) \times MC(B) \longrightarrow MC(A \otimes_\infty B).$$

PROOF. We need to prove that if $x_0 \sim x_1 \in \widehat{MC}(A)$ and $y_0 \sim y_1 \in \widehat{MC}(B)$ then $x_0 \boxtimes y_0 \sim x_1 \boxtimes y_1$. We will do this in two steps showing that $x_0 \boxtimes y_0 \sim x_1 \boxtimes y_0 \sim x_1 \boxtimes y_1$.

For the first equivalence we observe that the A_∞ -algebra $(A \otimes I) \otimes B$ together with the maps $p'_j = p_j \otimes id : (A \otimes I) \otimes B \longrightarrow A \otimes B$ and $i' = i \otimes id : A \otimes B \longrightarrow (A \otimes I) \otimes B$ is another model for $(A \otimes B)^{[0,1]}$. Now by definition there is $c \in \widehat{MC}(A \otimes I)$ such that $p_j(c) = x_j$. Then, by Proposition 6.2, $c \boxtimes y_0 \in \widehat{MC}((A \otimes I) \otimes B)$ and we have $p'_j(c \boxtimes y_0) = x_j \boxtimes y_0$. Therefore $x_0 \boxtimes y_0 \sim x_1 \boxtimes y_0$, by definition.

Similarly, using $A \otimes (B \otimes I)$ as a model for $(A \otimes B)^{[0,1]}$, we see that $x_1 \boxtimes y_0 \sim x_1 \boxtimes y_1$. \square

In general, the map \boxtimes is neither injective nor surjective. However, when A and B are graded, there is one simple situation where we can show that \boxtimes is a bijection.

Definition 6.7. A filtered A_∞ -algebra (A, \mathfrak{m}) is connected if $H^0(A, \mathfrak{m}_{1,0}) = \mathbb{K}$.

Proposition 6.8. *Let A and B be graded, connected A_∞ -algebras. Then $\boxtimes : MC(A) \times MC(B) \longrightarrow MC(A \otimes_\infty B)$ is a bijection.*

PROOF. Firstly we note that, by replacing A and B by their canonical models we can assume that $\mathfrak{m}_{1,0} = 0$ and thus $A^0 = \mathbb{K}e_A$ and $B^0 = \mathbb{K}e_B$. Secondly, we can ignore gauge equivalence since equation (31) and unitality imply this is a trivial relation in this situation.

We now proceed by direct computation. From the definition of grading on the tensor product

$$(A \otimes B)^1 = e_A \otimes B^1 \oplus A^1 \otimes e_B.$$

Thus any element $z \in (A \otimes B)^1 \hat{\otimes} \Lambda_0$ is of the form $z = x \otimes e_B + e_A \otimes y = x \boxtimes y$. As in the proof of Proposition 6.2, we have

$$\sum_n \mathfrak{m}_n^\otimes(z, \dots, z) = \left[\sum_n \mathfrak{m}_n(x, \dots, x) \right] \otimes e_B + e_A \otimes \left[\sum_n \mathfrak{m}_n(y, \dots, y) \right].$$

Hence $z \in MC(A \otimes_\infty B)$ if and only if $x \in MC(A)$ and $y \in MC(B)$. \square

6.2 Cohomology and deformations of the tensor product

As we saw in Section 2, each bounding cochain $x \in MC(A)$ determines a deformation of the A_∞ -algebra A . In this section we will describe this deformation for the bounding cochains $x \boxtimes y \in MC(A \otimes_\infty B)$ constructed in the previous subsection.

To avoid working with completions we will assume that all the A_∞ -algebras are *compact*. We say an A_∞ -algebra is compact if $H^*(A, \mathfrak{m}_{1,0})$ is a finite dimensional \mathbb{K} -vector space. By replacing the A_∞ -algebra by its canonical model, if necessary, this is equivalent to assuming that A is finite dimensional. Under this assumption, the tensor product $A \otimes_{\mathbb{K}} \Lambda_0$ is already complete, therefore

$$\hat{A}_0 = A \otimes_{\mathbb{K}} \Lambda_0 \text{ and } \hat{A} = A \otimes_{\mathbb{K}} \Lambda.$$

Similarly $\widehat{A \otimes B}_0 = \hat{A}_0 \otimes_{\Lambda_0} \hat{B}_0$ and $\widehat{A \otimes B} = \hat{A} \otimes_{\Lambda} \hat{B}$.

Under these assumptions, we will show that $(\widehat{A \otimes B}, \mathfrak{m}_k^{\otimes, x \boxtimes y})$ is quasi-isomorphic to $(\hat{A}, \mathfrak{m}_k^x) \otimes_\infty (\hat{B}, \mathfrak{m}_k^y)$, as classical A_∞ -algebras over Λ . This will be a direct consequence of Theorem 5.3, once we prove the following

Proposition 6.9. *Let $x \in MC(A)$ and $y \in MC(B)$ be bounding cochains and consider $z = x \boxtimes y \in MC(A \otimes_\infty B)$. Then $(\hat{A}, \mathfrak{m}_k^x)$ and $(\hat{B}, \mathfrak{m}_k^y)$ are commuting subalgebras of $(\hat{A} \otimes_{\Lambda} \hat{B}, \mathfrak{m}_k^{\otimes, z})$, under the inclusions*

$$\begin{aligned} \hat{A} &\longrightarrow \hat{A} \otimes_{\Lambda} \hat{B}, & \hat{B} &\longrightarrow \hat{A} \otimes_{\Lambda} \hat{B} \\ a &\longrightarrow a \otimes e_B & b &\longrightarrow e_A \otimes b. \end{aligned}$$

PROOF. First we need to prove that \hat{A} is a subalgebra, that is, for each $n > 0$ and $a_1, \dots, a_n \in \hat{A}$ we need to check

$$\mathfrak{m}_n^{\otimes, z}(a_1, \dots, a_n) = \mathfrak{m}_n^x(a_1, \dots, a_n).$$

Using the same argument as in Proposition 6.2 we obtain

$$\mathfrak{m}_n^{\otimes, z}(a_1, \dots, a_n) = \sum_{k=i_0+\dots+i_n} \mathfrak{m}_{n+k}(x, \dots, x, a_1, x, \dots, a_n, x, \dots, x) \otimes e_B = \mathfrak{m}_n^x(a_1, \dots, a_n).$$

Analogously we see that B is a subalgebra.

Next we check the commuting relations in Definition 5.2. Consider $c, c_1, \dots, c_n \in \hat{A} \otimes_{\Lambda} \hat{B}$, such that, for each i , $c_i = a_i \otimes e_B$ or $c_i = e_A \otimes b_i$ and $c = a \otimes b$. Again, repeating the same argument we conclude that

$$\mathfrak{m}_n^{\otimes, z}(c_1, \dots, c_n) = 0$$

unless, all the $c_i = a_i \otimes e_B$, or all the $c_i = e_A \otimes b_i$, or if $n = 2$. In the latter case, there are two possibilities, we check only one of them, namely $c_1 = a_1 \otimes e_B$ and $c_2 = e_A \otimes b_2$. We compute

$$\mathfrak{m}_2^{\otimes, z}(c_1, c_2) = \mathfrak{m}_2(a_1, e_A) \otimes \mathfrak{m}_2(e_B, b_2) = (-1)^{|a_1|} a_1 \otimes b_2,$$

$$\mathfrak{m}_2^{\otimes, z}(c_2, c_1) = (-1)^{|a_1||b_2|} \mathfrak{m}_2(e_A, a_1) \otimes \mathfrak{m}_2(b_2, e_B) = (-1)^{|b_2|+|a_1||b_2|} a_1 \otimes b_2.$$

Therefore we conclude $\mathfrak{m}_2^{\otimes, z}(c_1, c_2) + (-1)^{||c_1|| ||c_2||} \mathfrak{m}_2^{\otimes, z}(c_2, c_1) = 0$.

Finally we compute

$$\begin{aligned} \mathfrak{m}_{n+1}^{\otimes, z}(c_1, \dots, c_i, c, c_{i+1}, \dots, c_n) &= \sum_{k=i_0+\dots+i_n} \mathfrak{m}_{n+k+1}^{\otimes}(z, \dots, z, c_1, z, \dots, c_n, z, \dots, z) \\ &= \sum_{k=i_0+\dots+i_n} \sum_{U, W} \pm U^A \otimes W^B(z, \dots, z, c_1, z, \dots, c_n, z, \dots, z). \end{aligned}$$

This sum can be further expanded since $z = x \otimes e_B + e_A \otimes y$. In the notation of Lemma 6.1, we have $k+n = j_{a,x} + j_{b,y}$, therefore the only nontrivial contributions come from trees U, W that satisfy

$$k+n = j_{a,x} + j_{b,y} \leq b(U) + b(W) \leq n+k+1.$$

Hence there are two possibilities: U is binary and W has a single internal vertex, or vice-versa. When $n=0$, both cases contribute and we have

$$\mathfrak{m}_1^{\otimes, z}(a \otimes b) = \mathfrak{m}_1^x(a) \otimes b + (-1)^{|a|} a \otimes \mathfrak{m}_1^y(b).$$

When $n \geq 1$, since A and B are unital, the only contributions come from the first case, when all the $c_i = e_A \otimes b_i$, or the second, when all the $c_i = a_i \otimes e_B$. A simple computation, using the formula for the signs in Theorem 4.1 shows that in the first case we get

$$\mathfrak{m}_{n+1}^{\otimes, z}(c_1, \dots, c_i, c, c_{i+1}, \dots, c_n) = (-1)^{|a|(1+\sum_{j \leq i} ||b_j||)} a \otimes \mathfrak{m}_{n+1}^y(b_1, \dots, b_i, b, \dots, b_n)$$

and in the second,

$$\mathfrak{m}_{n+1}^{\otimes, z}(c_1, \dots, c_i, c, c_{i+1}, \dots, c_n) = (-1)^{|b|(\sum_{i+1 \leq j} ||a_j||)} \mathfrak{m}_{n+1}^x(a_1, \dots, a_i, a, \dots, a_n) \otimes b,$$

as required. \square

Corollary 6.10. *Let $x \in MC(A)$ and $y \in MC(B)$ be bounding cochains and consider the bounding cochain $z = x \boxtimes y \in MC(A \otimes_\infty B)$. We have the isomorphism of classical A_∞ -algebras over Λ ,*

$$(\hat{A}, \mathfrak{m}^x) \otimes_\infty (\hat{B}, \mathfrak{m}^y) \simeq (\widehat{A \otimes_\infty B}, \mathfrak{m}^{\otimes, z}).$$

PROOF. Proposition 6.9 and Theorem 5.3 immediately imply the result, since $K : \hat{A} \otimes_\Lambda \hat{B} \longrightarrow \widehat{A \otimes_\infty B}$ is simply the identity. \square

Proposition 6.9 has one additional consequence.

Corollary 6.11. *Consider $x \in MC(A)$ and $y \in MC(B)$. For each $n \in \mathbb{Z}_2$, we have the exact sequence*

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i+j=n} H^i(A, x; \Lambda_0) \otimes H^j(B, y; \Lambda_0) &\longrightarrow H^n(A \otimes_\infty B, x \boxtimes y; \Lambda_0) \\ &\longrightarrow \bigoplus_{i+j=n-1} \text{Tor}_1^{\Lambda_0}(H^i(A, x; \Lambda_0), H^j(B, y; \Lambda_0)) \longrightarrow 0. \end{aligned}$$

PROOF. During the proof of Proposition 6.9 we saw that $\mathfrak{m}_1^{\otimes, x \boxtimes y} = \mathfrak{m}_1^x \otimes id + id \otimes \mathfrak{m}_1^y$. The result then follows from the usual Künneth formula [15, Theorem 3.6.1], once we prove that \hat{A}^i and $\mathfrak{m}_1^x(\hat{A}^i)$ are flat Λ_0 -modules, for $i \in \mathbb{Z}_2$. The module \hat{A}^i is free by assumption and $\mathfrak{m}_1^x(\hat{A}^i)$ is a finitely generated submodule of \hat{A}^{i+1} , therefore it is also free by Corollary 2.6.7 in [5]. \square

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